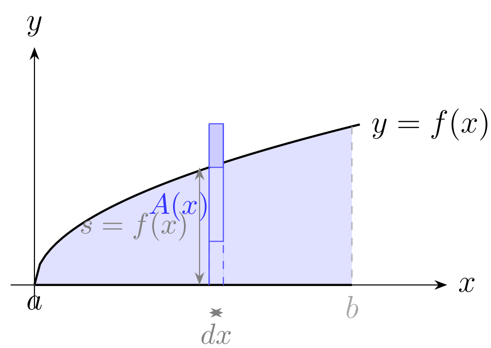
Volumes with Known Cross-Sections
ApexMath

Notes

The Big Idea: Building a Solid from a Base

In the Disk and Washer method, the cross-sections of the solid were always circles or washers because we were rotating a region. Now we look at a different kind of solid.

Here, a 2D region in the xy-plane acts as the *base* of a solid. At every x-value across that base, a flat shape — called a **cross-section** — stands straight up, perpendicular to the x-axis. The cross-section can be a square, rectangle, semicircle, equilateral triangle, or any other specified shape. As x moves from a to b, these cross-sections sweep out a 3D solid.



The volume of one thin slab is:

$$dV = A(x)\,dx$$

where $A(x)$ is the area of the cross-section at position x, and dx is the thickness of the slab. The total volume is:

$$\boxed{V = \int_a^b A(x)\,dx}$$

This is the same strip strategy used throughout the course. The only new skill is computing $A(x)$ from the geometry of the cross-section.

How to Find $A(x)$ for Each Shape

The cross-section always sits on top of the base region. Its size depends on the *side length* or *diameter* of the shape at each x-value. That length comes directly from the base region.

If the base is the region between $y = f(x)$ (top) and $y = g(x)$ (bottom), then the width of the base at position x is:

$$s(x) = f(x) - g(x)$$

This width $s(x)$ is what you plug into the area formula for your shape.

Square cross-sections (side length = s):

$$A(x) = s^2 = [f(x) - g(x)]^2$$

Rectangle cross-sections (base = s, height = ks for some constant k):

$$A(x) = k \cdot s^2 = k \cdot [f(x) - g(x)]^2$$

Semicircle cross-sections (diameter = s, so radius = $s/2$):

$$A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\frac{s}{2}\right)^2 = \frac{\pi}{8}\,[f(x) - g(x)]^2$$

Equilateral triangle cross-sections (side length = s):

$$A(x) = \frac{\sqrt{3}}{4}\,s^2 = \frac{\sqrt{3}}{4}\,[f(x) - g(x)]^2$$

Pro Tip

- **Always identify $s(x)$ first.** Before writing $A(x)$, clearly state what $s(x)$ is. It is always the width of the base at position x, which is $f(x) - g(x)$.

- **Write out $A(x)$ on its own line.** After finding $s(x)$, substitute it into the correct area formula and simplify fully before setting up the integral. Trying to do everything at once leads to errors.

- **Do not confuse diameter and radius for semicircles.** If the problem says the diameter equals $s(x)$, then the radius is $\frac{s(x)}{2}$. The area formula is $\frac{1}{2}\pi r^2$, so you must halve $s(x)$ *before* squaring.

- **Read the problem carefully for what equals $s(x)$.** Some problems define the cross-section side as $f(x)$ alone (when the base region sits on the x-axis and $g(x) = 0$). Others use $f(x) - g(x)$ when the base is between two curves. Identify which applies.

- **The formula $V = \int_a^b A(x)\,dx$ always works.** No matter what the cross-section shape is, the strategy is always the same: find $A(x)$, then integrate.

Examples

Example 1 (Square Cross-Sections): The base of a solid is the region between $y = \sqrt{x}$ and $y = 0$ from $x = 0$ to $x = 4$. Cross-sections perpendicular to the x-axis are squares. Find the volume.

Step 1: Identify $s(x)$.
The top curve is $\sqrt{x}$ and the bottom is 0, so:

$$s(x) = \sqrt{x} - 0 = \sqrt{x}$$

Step 2: Write $A(x)$ for a square.
A square with side s has area s^2:

$$A(x) = [\sqrt{x}]^2 = x$$

Step 3: Set up and evaluate the integral.

$$V = \int_0^4 A(x)\,dx = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{16}{2} - 0$$

$$\boxed{V = 8}$$

Example 2 (Semicircle Cross-Sections): The base of a solid is the region between $y = x^2$ and $y = 0$ from $x = 0$ to $x = 2$. Cross-sections perpendicular to the x-axis are semicircles with diameter equal to $s(x)$. Find the volume.

Step 1: Identify $s(x)$.
The top curve is x^2 and the bottom is 0:

$$s(x) = x^2 - 0 = x^2$$

Step 2: Find the radius.
The diameter equals $s(x) = x^2$, so the radius is:

$$r(x) = \frac{x^2}{2}$$

Step 3: Write $A(x)$ for a semicircle.
The area of a semicircle is $\frac{1}{2}\pi r^2$:

$$A(x) = \frac{1}{2}\pi\left(\frac{x^2}{2}\right)^2 = \frac{1}{2}\pi \cdot \frac{x^4}{4} = \frac{\pi x^4}{8}$$

Step 4: Set up and evaluate the integral.

$$V = \int_0^2 \frac{\pi x^4}{8}\,dx = \frac{\pi}{8}\left[\frac{x^5}{5}\right]_0^2 = \frac{\pi}{8} \cdot \frac{32}{5} = \frac{32\pi}{40}$$

$$\boxed{V = \frac{4\pi}{5}}$$

Example 3 (Square Cross-Sections, Two Curves): The base of a solid is the region between $f(x) = x$ and $g(x) = x^2$ from $x = 0$ to $x = 1$. Cross-sections perpendicular to the x-axis are squares. Find the volume.

Step 1: Identify which curve is on top.
Test $x = 0.5$: $f(0.5) = 0.5 > g(0.5) = 0.25$. So $f(x) = x$ is on top.
Step 2: Find $s(x)$.

$$s(x) = f(x) - g(x) = x - x^2$$

Step 3: Write $A(x)$ for a square.

$$A(x) = [s(x)]^2 = (x - x^2)^2 = x^2 - 2x^3 + x^4$$

Step 4: Set up and evaluate the integral.

$$V = \int_0^1 (x^2 - 2x^3 + x^4)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5}$$

Find a common denominator of 30:

$$V = \frac{10}{30} - \frac{15}{30} + \frac{6}{30} = \frac{1}{30}$$

$$\boxed{V = \frac{1}{30}}$$

Example 4 (Equilateral Triangle Cross-Sections): The base of a solid is the region between $y = 4 - x^2$ and $y = 0$ from $x = -2$ to $x = 2$. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume.

Step 1: Identify $s(x)$.
The top curve is $4 - x^2$ and the bottom is 0:

$$s(x) = 4 - x^2$$

Step 2: Write $A(x)$ for an equilateral triangle.

The area of an equilateral triangle with side s is $\frac{\sqrt{3}}{4}s^2$:

$$A(x) = \frac{\sqrt{3}}{4}(4 - x^2)^2 = \frac{\sqrt{3}}{4}(16 - 8x^2 + x^4)$$

Step 3: Set up and evaluate the integral.

$$V = \int_{-2}^{2} \frac{\sqrt{3}}{4}(16 - 8x^2 + x^4)\, dx = \frac{\sqrt{3}}{4}\int_{-2}^{2}(16 - 8x^2 + x^4)\, dx$$

Since $16 - 8x^2 + x^4$ is an even function, we can use symmetry:

$$V = \frac{\sqrt{3}}{4} \cdot 2\int_0^2 (16 - 8x^2 + x^4)\, dx = \frac{\sqrt{3}}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$$

Evaluate at $x = 2$:

$$16(2) - \frac{8(8)}{3} + \frac{32}{5} = 32 - \frac{64}{3} + \frac{32}{5}$$

Common denominator of 15:

$$= \frac{480}{15} - \frac{320}{15} + \frac{96}{15} = \frac{256}{15}$$

Therefore:

$$V = \frac{\sqrt{3}}{2} \cdot \frac{256}{15} = \frac{256\sqrt{3}}{30}$$

$$\boxed{V = \frac{128\sqrt{3}}{15}}$$

Common Mistakes

- **Plugging $f(x)$ directly into the area formula without finding $s(x)$:** The side length of the cross-section is the *width* of the base at position x, which is $f(x) - g(x)$, not just $f(x)$ alone. If the base touches the x-axis then $g(x) = 0$ and $s(x) = f(x)$, but this should be stated explicitly.
- **Using diameter instead of radius for semicircles:** The area formula $\frac{1}{2}\pi r^2$ requires the radius. If the problem gives you the diameter $s(x)$, you must compute $r = \frac{s(x)}{2}$ and then square r, not s.
- **Forgetting to square $s(x)$:** Every shape in this lesson has area proportional to s^2. Writing $A(x) = s(x)$ instead of $A(x) = [s(x)]^2$ (or the appropriate multiple) is a very common error.
- **Mixing up cross-section area formulas:** Keep these straight — square: s^2; semicircle: $\frac{\pi}{8}s^2$; equilateral triangle: $\frac{\sqrt{3}}{4}s^2$. Write the formula explicitly before substituting.
- **Forgetting to find intersection points when the bounds are not given:** If the problem describes the region as "enclosed by" two curves without stating a and b, you must solve $f(x) = g(x)$ to find the limits of integration first.

Worksheet

1. The base of a solid is the region between $y = x$ and $y = 0$ from $x = 0$ to $x = 3$. Cross-sections perpendicular to the x-axis are squares. Find the volume.

2. The base of a solid is the region between $y = \cos x$ and $y = 0$ from $x = 0$ to $x = \dfrac{\pi}{2}$. Cross-sections perpendicular to the x-axis are squares. Find the volume.

3. The base of a solid is the region between $y = 4 - x^2$ and $y = 0$ from $x = -2$ to $x = 2$. Cross-sections perpendicular to the x-axis are semicircles with diameter equal to $s(x)$. Find the volume.

4. The base of a solid is the region enclosed by $f(x) = \sqrt{x}$ and $g(x) = x$ from $x = 0$ to $x = 1$. Cross-sections perpendicular to the x-axis are squares. Find the volume.

5. The base of a solid is the region between $y = e^x$ and $y = 0$ from $x = 0$ to $x = 1$. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume.

6. The base of a solid is the region enclosed by $f(x) = 2\sqrt{x}$ and $g(x) = 2x$ from $x = 0$ to $x = 1$. Cross-sections perpendicular to the x-axis are semicircles with diameter equal to $s(x)$. Find the volume.

7. The base of a solid is the region between $y = \sin x$ and $y = 0$ from $x = 0$ to $x = \pi$. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume.

8. (Challenge) The base of a solid is the region enclosed by $f(x) = x^2$ and $g(x) = 4$. Cross-sections perpendicular to the x-axis are rectangles whose height is twice the base. Find the volume.

Answer Key

1. Base: $y = x$ and $y = 0$, $x \in [0, 3]$. Square cross-sections.

Step 1: Identify $s(x)$.

$$s(x) = x - 0 = x$$

Step 2: Write $A(x)$ for a square.

$$A(x) = [s(x)]^2 = x^2$$

Step 3: Integrate.

$$V = \int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0$$

$$\boxed{V = 9}$$

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2. Base: $y = \cos x$ and $y = 0$, $x \in \left[0, \frac{\pi}{2}\right]$. Square cross-sections.

Step 1: Identify $s(x)$.

$$s(x) = \cos x - 0 = \cos x$$

Step 2: Write $A(x)$ for a square.

$$A(x) = \cos^2 x$$

Step 3: Apply the identity $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ to make the integral manageable.

$$V = \int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1 + \cos 2x}{2}\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

At $x = \frac{\pi}{2}$: $\frac{\pi}{2} + \frac{\sin \pi}{2} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$
At $x = 0$: $0 + 0 = 0$

$$V = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{V = \frac{\pi}{4}}$$

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3. Base: $y = 4 - x^2$ and $y = 0$, $x \in [-2, 2]$. Semicircle cross-sections, diameter $= s(x)$.

Step 1: Identify $s(x)$.

$$s(x) = 4 - x^2$$

Step 2: Find the radius.

$$r(x) = \frac{4 - x^2}{2}$$

Step 3: Write $A(x)$ for a semicircle.

$$A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\frac{4 - x^2}{2}\right)^2 = \frac{\pi}{2} \cdot \frac{(4 - x^2)^2}{4} = \frac{\pi(4 - x^2)^2}{8}$$

Step 4: Expand $(4 - x^2)^2$.

$$(4 - x^2)^2 = 16 - 8x^2 + x^4$$

Step 5: Integrate. Since the integrand is even, use symmetry.

$$V = \frac{\pi}{8}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{\pi}{8} \cdot 2\int_{0}^{2}(16 - 8x^2 + x^4)\,dx = \frac{\pi}{4}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$$

At $x = 2$:

$$32 - \frac{64}{3} + \frac{32}{5} = \frac{480 - 320 + 96}{15} = \frac{256}{15}$$

$$V = \frac{\pi}{4} \cdot \frac{256}{15} = \frac{256\pi}{60}$$

$$\boxed{V = \frac{64\pi}{15}}$$

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4. Base: $f(x) = \sqrt{x}$ and $g(x) = x$, $x \in [0, 1]$. Square cross-sections.

Step 1: Identify which curve is on top.
Test $x = 0.25$: $\sqrt{0.25} = 0.5 > 0.25$. So $\sqrt{x}$ is on top.
Step 2: Find $s(x)$.

$$s(x) = \sqrt{x} - x$$

Step 3: Write $A(x)$ for a square.

$$A(x) = (\sqrt{x} - x)^2 = x - 2x^{3/2} + x^2$$

Step 4: Integrate.

$$V = \int_0^1 (x - 2x^{3/2} + x^2)\,dx = \left[\frac{x^2}{2} - 2 \cdot \frac{x^{5/2}}{5/2} + \frac{x^3}{3}\right]_0^1 = \left[\frac{x^2}{2} - \frac{4x^{5/2}}{5} + \frac{x^3}{3}\right]_0^1$$

At $x = 1$:

$$\frac{1}{2} - \frac{4}{5} + \frac{1}{3} = \frac{15}{30} - \frac{24}{30} + \frac{10}{30} = \frac{1}{30}$$

$$\boxed{V = \frac{1}{30}}$$

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5. Base: $y = e^x$ and $y = 0$, $x \in [0, 1]$. Equilateral triangle cross-sections.

Step 1: Identify $s(x)$.

$$s(x) = e^x - 0 = e^x$$

Step 2: Write $A(x)$ for an equilateral triangle.

$$A(x) = \frac{\sqrt{3}}{4}[e^x]^2 = \frac{\sqrt{3}}{4}e^{2x}$$

Step 3: Integrate.

$$V = \int_0^1 \frac{\sqrt{3}}{4}e^{2x}\,dx = \frac{\sqrt{3}}{4}\left[\frac{e^{2x}}{2}\right]_0^1 = \frac{\sqrt{3}}{8}\left[e^2 - e^0\right] = \frac{\sqrt{3}}{8}(e^2 - 1)$$

$$\boxed{V = \frac{\sqrt{3}}{8}(e^2 - 1)}$$

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6. Base: $f(x) = 2\sqrt{x}$ and $g(x) = 2x$, $x \in [0, 1]$. Semicircle cross-sections, diameter $= s(x)$.

Step 1: Identify which curve is on top.
Test $x = 0.25$: $2\sqrt{0.25} = 1 > 2(0.25) = 0.5$. So $2\sqrt{x}$ is on top.
Step 2: Find $s(x)$.

$$s(x) = 2\sqrt{x} - 2x$$

Step 3: Find the radius.

$$r(x) = \frac{s(x)}{2} = \sqrt{x} - x$$

Step 4: Write $A(x)$ for a semicircle.

$$A(x) = \frac{\pi}{2}r^2 = \frac{\pi}{2}(\sqrt{x} - x)^2 = \frac{\pi}{2}(x - 2x^{3/2} + x^2)$$

Step 5: Integrate.

$$V = \frac{\pi}{2}\int_0^1 (x - 2x^{3/2} + x^2)\,dx = \frac{\pi}{2}\left[\frac{x^2}{2} - \frac{4x^{5/2}}{5} + \frac{x^3}{3}\right]_0^1$$

At $x = 1$:

$$\frac{1}{2} - \frac{4}{5} + \frac{1}{3} = \frac{15 - 24 + 10}{30} = \frac{1}{30}$$

$$V = \frac{\pi}{2} \cdot \frac{1}{30}$$

$$\boxed{V = \frac{\pi}{60}}$$

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7. Base: $y = \sin x$ and $y = 0$, $x \in [0, \pi]$. Equilateral triangle cross-sections.

Step 1: Identify $s(x)$.

$$s(x) = \sin x$$

Step 2: Write $A(x)$ for an equilateral triangle.

$$A(x) = \frac{\sqrt{3}}{4} \sin^2 x$$

Step 3: Apply the identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

$$V = \frac{\sqrt{3}}{4} \int_0^{\pi} \sin^2 x \, dx = \frac{\sqrt{3}}{4} \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx = \frac{\sqrt{3}}{8} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

At $x = \pi$: $\pi - \dfrac{\sin 2\pi}{2} = \pi - 0 = \pi$
At $x = 0$: $0 - 0 = 0$

$$V = \frac{\sqrt{3}}{8} \cdot \pi$$

$$\boxed{V = \frac{\sqrt{3}\,\pi}{8}}$$

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8. (Challenge) Base: $f(x) = x^2$ and $g(x) = 4$. Rectangle cross-sections with height $= 2 \times$ base.

Step 1: Find intersection points.

$$x^2 = 4 \implies x = \pm 2$$

Step 2: Identify which curve is on top on $[-2, 2]$.
Test $x = 0$: $g(0) = 4 > f(0) = 0$. So $g(x) = 4$ is on top.
Step 3: Find $s(x)$ (the base of the rectangle).

$$s(x) = 4 - x^2$$

Step 4: Write $A(x)$ for the rectangle.
The height is twice the base, so height $= 2s(x)$:

$$A(x) = \text{base} \times \text{height} = s(x) \cdot 2s(x) = 2[s(x)]^2 = 2(4 - x^2)^2$$

Step 5: Expand $(4-x^2)^2$ and integrate.

$$(4-x^2)^2 = 16 - 8x^2 + x^4$$

$$V = 2\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx = 2 \cdot 2\int_{0}^{2} (16 - 8x^2 + x^4)\,dx = 4\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$$

At $x = 2$: $\dfrac{480 - 320 + 96}{15} = \dfrac{256}{15}$

$$V = 4 \cdot \frac{256}{15} = \frac{1024}{15}$$

$$\boxed{V = \frac{1024}{15}}$$

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End of Lesson • ApexMath
