Volumes of Revolution
Disk and Washer Method
ApexMath

Notes

The Big Idea: Rotating a Region

When you rotate a 2D region around an axis, it sweeps out a 3D solid. To find the volume of that solid, we use the same strip strategy from the area lessons. We slice the solid into thin pieces, find the volume dV of one thin piece, then integrate to get the total volume V.

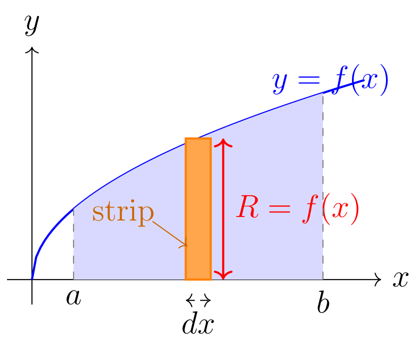
$$V = \int dV$$

The Disk Method

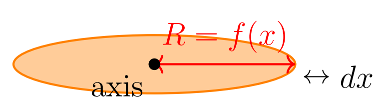
When a region between a curve $y = f(x)$ and the x-axis is rotated around the x-axis, each thin vertical strip sweeps out a **disk** (a solid circle). The disk has:

- Radius: $R = f(x)$ (the height of the strip)
- Thickness: dx
- Volume of one disk: $dV = \pi R^2\, dx = \pi[f(x)]^2\, dx$

$$\boxed{V = \pi \int_a^b [f(x)]^2\, dx}$$



Rotate around x-axis $\longrightarrow$
$dV = \pi[f(x)]^2\, dx$



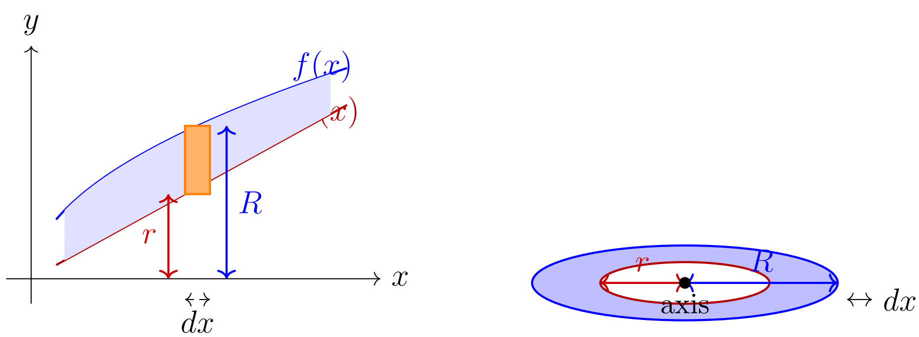
Disk cross-section
$dV = \pi R^2\, dx$

The Washer Method

When the region is between *two* curves $y = f(x)$ (outer) and $y = g(x)$ (inner) and both are rotated around the x-axis, each strip sweeps out a **washer** (a disk with a hole in the middle). The washer has:

- Outer radius: $R = f(x)$
- Inner radius: $r = g(x)$
- Volume of one washer: $dV = \pi(R^2 - r^2)\,dx = \pi\left([f(x)]^2 - [g(x)]^2\right)dx$

$$\boxed{V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx}$$



Rotate around x-axis $\longrightarrow$
$dV = \pi(R^2 - r^2)\,dx$

Washer cross-section
$dV = \pi(R^2 - r^2)\,dx$

Rotating Around a Different Axis

When the rotation axis is not the x-axis but a horizontal line like $y = k$, the radii must be measured from that line, not from the x-axis:

- Outer radius: $R = f(x) - k \quad (\text{if } f(x) > k)$
- Inner radius: $r = g(x) - k \quad (\text{if } g(x) > k)$

If the axis is *below* the region (e.g. $y = -1$), then distances are larger and $R = f(x) - (-1) = f(x) + 1$.

Rotating around the y-axis: Use horizontal strips. Express the curves as $x = h(y)$ and integrate with respect to y:

$$V = \pi \int_c^d [h(y)]^2\,dy \quad (\text{disk}) \qquad V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2\right)dy \quad (\text{washer})$$

Pro Tip

- Always write dV before integrating. Identify whether each slice is a disk (one curve) or a washer (two curves), then write the radius or radii explicitly.
- For the washer method, R is always the curve farther from the axis and r is the curve closer to the axis. If you mix them up, you get a negative value inside the parentheses.

- When rotating around $y = k$, the radius of a point $(x, f(x))$ from the line $y = k$ is $|f(x) - k|$. Write this out carefully before squaring.
- Do not forget to include π in the formula. It is part of the disk/washer area and cannot be dropped.
- A quick sanity check: the volume of a disk/washer must be positive. If your answer is negative, you have subtracted in the wrong order inside the integral.

Examples

Example 1 (Disk Method — x-axis): Find the volume of the solid formed by rotating $y = \sqrt{x}$ from $x = 0$ to $x = 4$ around the x-axis.

Step 1: Identify the strip and write dV.
Each vertical strip sweeps out a disk with radius $R = \sqrt{x}$ and thickness dx.

$$dV = \pi R^2\,dx = \pi(\sqrt{x})^2\,dx = \pi x\,dx$$

Step 2: Integrate.

$$V = \int dV = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot 8$$

$$\boxed{V = 8\pi}$$

Example 2 (Disk Method — y-axis): Find the volume of the solid formed by rotating $y = x^2$ from $y = 0$ to $y = 4$ around the y-axis.

Step 1: Rewrite $y = x^2$ as $x = \sqrt{y}$ (solve for x since we integrate in y).
Each horizontal strip sweeps out a disk with radius $R = \sqrt{y}$ and thickness dy.

$$dV = \pi R^2\,dy = \pi(\sqrt{y})^2\,dy = \pi y\,dy$$

Step 2: Integrate from $y = 0$ to $y = 4$.

$$V = \pi \int_0^4 y\,dy = \pi \left[\frac{y^2}{2}\right]_0^4 = \pi \cdot 8$$

$$\boxed{V = 8\pi}$$

Example 3 (Washer Method — x-axis): Find the volume of the solid formed by rotating the region between $f(x) = \sqrt{x}$ and $g(x) = x^2$ from $x = 0$ to $x = 1$ around the x-axis.

Step 1: Identify outer and inner radii.
Test $x = 0.5$: $\sqrt{0.5} \approx 0.707 > 0.25 = (0.5)^2$. So $f(x) = \sqrt{x}$ is farther from the axis.
$R = \sqrt{x}$ (outer), $r = x^2$ (inner).
Step 2: Write dV.

$$dV = \pi(R^2 - r^2)\,dx = \pi\left[(\sqrt{x})^2 - (x^2)^2\right]dx = \pi(x - x^4)\,dx$$

Step 3: Integrate.

$$V = \pi\int_0^1 (x - x^4)\,dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \pi \cdot \frac{3}{10}$$

$$\boxed{V = \frac{3\pi}{10}}$$

Example 4 (Washer — Rotated Around $y = -1$): Find the volume of the solid formed by rotating $f(x) = x^2$ from $x = 0$ to $x = 2$ around the line $y = -1$.

Step 1: The axis is $y = -1$, which is below the region. Measure radius from $y = -1$.
The radius of the strip at position x is the distance from the curve to the axis:

$$R = f(x) - (-1) = x^2 + 1$$

There is no inner curve, so this is a disk method with the shifted radius.
Step 2: Write dV.

$$dV = \pi R^2\,dx = \pi(x^2 + 1)^2\,dx = \pi(x^4 + 2x^2 + 1)\,dx$$

Step 3: Integrate from $x = 0$ to $x = 2$.

$$V = \pi\int_0^2 (x^4 + 2x^2 + 1)\,dx = \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^2$$

$$= \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right) = \pi \cdot \frac{96 + 80 + 30}{15} = \pi \cdot \frac{206}{15}$$

$$\boxed{V = \frac{206\pi}{15}}$$

Example 5 (Washer — Finding Intersections First): Find the volume of the solid formed by rotating the region enclosed by $f(x) = 4 - x^2$ and $g(x) = 3$ around the x-axis.

Step 1: Find intersection points.

$$4 - x^2 = 3 \implies x^2 = 1 \implies x = \pm 1$$

Step 2: Identify outer and inner radii on $[-1, 1]$.
Test $x = 0$: $f(0) = 4 > g(0) = 3$. So f is the outer curve.
$R = 4 - x^2$, $r = 3$.
Step 3: Write dV.

$$dV = \pi(R^2 - r^2)\, dx = \pi \left[(4 - x^2)^2 - 9\right] dx$$

$$= \pi \left[16 - 8x^2 + x^4 - 9\right] dx = \pi(7 - 8x^2 + x^4)\, dx$$

Step 4: Integrate from $x = -1$ to $x = 1$.

$$V = \pi \int_{-1}^{1} (7 - 8x^2 + x^4)\, dx = \pi \left[7x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1}$$

At $x = 1$: $7 - \frac{8}{3} + \frac{1}{5} = \frac{105 - 40 + 3}{15} = \frac{68}{15}$
At $x = -1$: $-7 + \frac{8}{3} - \frac{1}{5} = -\frac{68}{15}$

$$V = \pi \left(\frac{68}{15} - \left(-\frac{68}{15}\right)\right) = \pi \cdot \frac{136}{15}$$

$$\boxed{V = \frac{136\pi}{15}}$$

Common Mistakes

- **Forgetting to square the radius:** The disk formula is πR^2, not πR. Since we are computing area of a circle cross-section, the radius must be squared. Forgetting to square gives the wrong formula entirely.

- **Subtracting radii instead of squared radii:** The washer formula is $\pi(R^2 - r^2)$, not $\pi(R - r)^2$. These are very different. Always square each radius separately, then subtract.

- **Getting outer and inner radii mixed up:** The outer radius R is the curve farther from the axis. If $R < r$, you have them backwards and will get a negative volume.

- **Forgetting to adjust the radius when rotating around a shifted axis:** When the axis is $y = k$ instead of $y = 0$, the radius is $f(x) - k$, not just $f(x)$. Always measure the radius as the distance from the curve to the axis.

- **Dropping the π:** The π factor comes from the area of a circle and is part of every disk and washer formula. It cannot be added back in at the end as an afterthought — it must be inside the integral setup from the start.

Worksheet

1. Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 3$ around the x-axis.

2. Find the volume of the solid formed by rotating $y = \sqrt{x}$ from $x = 1$ to $x = 4$ around the x-axis.

3. Find the volume of the solid formed by rotating $y = x^3$ from $y = 0$ to $y = 8$ around the y-axis.

   (Rewrite as $x = y^{1/3}$ and integrate with respect to y.)

4. Find the volume of the solid formed by rotating the region between $f(x) = x$ and $g(x) = x^2$ from $x = 0$ to $x = 1$ around the x-axis.

5. Find the volume of the solid formed by rotating the region between $f(x) = 2$ and $g(x) = x^2$ from $x = -\sqrt{2}$ to $x = \sqrt{2}$ around the x-axis.

6. Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 2$ around the y-axis.

   (Rewrite as $x = \sqrt{y}$ and find appropriate y-limits.)

7. Find the volume of the solid formed by rotating $f(x) = x + 1$ from $x = 0$ to $x = 2$ around the line $y = -2$.

8. Find the volume of the solid formed by rotating the region between $f(x) = \sqrt{x}$ and $g(x) = 0$ from $x = 0$ to $x = 9$ around the line $y = 3$.

   (Axis is above the region. $R = 3 - 0 = 3$ and $r = 3 - \sqrt{x}$.)

9. (Challenge) Find the volume of the solid formed by rotating the region enclosed by $f(x) = x^2$ and $g(x) = 2x$ around the x-axis.

   (Find intersection points first.)

10. (Challenge) Find the volume of the solid formed by rotating $y = e^x$ from $x = 0$ to $x = 1$ around the x-axis.

Answer Key

1. $y = x^2$ around x-axis, $[0, 3]$.
$dV = \pi(x^2)^2\,dx = \pi x^4\,dx$

$$V = \pi \int_0^3 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^3 = \pi \cdot \frac{243}{5}$$

$$\boxed{V = \frac{243\pi}{5}}$$

2. $y = \sqrt{x}$ around x-axis, $[1, 4]$.
$dV = \pi(\sqrt{x})^2\,dx = \pi x\,dx$

$$V = \pi \int_1^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_1^4 = \pi\left(8 - \frac{1}{2}\right) = \frac{15\pi}{2}$$

$$\boxed{V = \frac{15\pi}{2}}$$

3. $y = x^3$ around y-axis, $y \in [0, 8]$.
Rewrite: $x = y^{1/3}$. $dV = \pi(y^{1/3})^2\,dy = \pi y^{2/3}\,dy$

$$V = \pi \int_0^8 y^{2/3}\,dy = \pi \left[\frac{y^{5/3}}{5/3}\right]_0^8 = \pi \cdot \frac{3}{5} \cdot 8^{5/3} = \pi \cdot \frac{3}{5} \cdot 32 = \frac{96\pi}{5}$$

$$\boxed{V = \frac{96\pi}{5}}$$

4. Region between $f(x) = x$ and $g(x) = x^2$ around x-axis, $[0, 1]$.
Test $x = 0.5$: $f = 0.5 > g = 0.25$. $R = x$, $r = x^2$.
$dV = \pi(x^2 - x^4)\,dx$

$$V = \pi \int_0^1 (x^2 - x^4)\,dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \pi \cdot \frac{2}{15}$$

$$\boxed{V = \frac{2\pi}{15}}$$

5. Region between $f(x) = 2$ and $g(x) = x^2$ around x-axis, $[-\sqrt{2}, \sqrt{2}]$.
$R = 2$, $r = x^2$.
$dV = \pi(4 - x^4)\,dx$

$$V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^4)\,dx = \pi \left[4x - \frac{x^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

At $x = \sqrt{2}$: $4\sqrt{2} - \frac{4\sqrt{2}}{5} = \sqrt{2}\left(4 - \frac{4}{5}\right) = \frac{16\sqrt{2}}{5}$
At $x = -\sqrt{2}$: $-\frac{16\sqrt{2}}{5}$

$$V = \pi \cdot \frac{32\sqrt{2}}{5}$$

$$\boxed{V = \frac{32\sqrt{2}\,\pi}{5}}$$

6. $y = x^2$ around y-axis, $x \in [0, 2]$.
y-limits: $y = 0$ to $y = 4$. Rewrite: $x = \sqrt{y}$. $dV = \pi y\, dy$

$$V = \pi \int_0^4 y\, dy = \pi \left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

$$\boxed{V = 8\pi}$$

7. $f(x) = x + 1$ around $y = -2$, $[0, 2]$.
Axis is $y = -2$, region is above it. $R = f(x) - (-2) = x + 1 + 2 = x + 3$.
$dV = \pi(x + 3)^2\, dx = \pi(x^2 + 6x + 9)\, dx$

$$V = \pi \int_0^2 (x^2 + 6x + 9)\, dx = \pi \left[\frac{x^3}{3} + 3x^2 + 9x\right]_0^2 = \pi\left(\frac{8}{3} + 12 + 18\right) = \pi \cdot \frac{8 + 90}{3} = \frac{98\pi}{3}$$

Wait — let me recompute: $\frac{8}{3} + 30 = \frac{8 + 90}{3} = \frac{98}{3}$.

$$\boxed{V = \frac{98\pi}{3}}$$

8. $f(x) = \sqrt{x}$ around $y = 3$, $[0, 9]$.
Axis is $y = 3$, which is *above* the curve $\sqrt{x}$ (since $\sqrt{x} \leq 3$ on $[0, 9]$).
Outer radius (from axis to x-axis): $R = 3 - 0 = 3$
Inner radius (from axis to curve): $r = 3 - \sqrt{x}$
$dV = \pi(R^2 - r^2)\, dx = \pi\left[9 - (3 - \sqrt{x})^2\right] dx = \pi\left[9 - (9 - 6\sqrt{x} + x)\right] dx = \pi(6\sqrt{x} - x)\, dx$

$$V = \pi \int_0^9 (6\sqrt{x} - x)\, dx = \pi\left[6 \cdot \frac{2x^{3/2}}{3} - \frac{x^2}{2}\right]_0^9 = \pi\left[4x^{3/2} - \frac{x^2}{2}\right]_0^9$$
$$= \pi\left(4 \cdot 27 - \frac{81}{2}\right) = \pi\left(108 - \frac{81}{2}\right) = \pi \cdot \frac{216 - 81}{2} = \frac{135\pi}{2}$$

$$\boxed{V = \frac{135\pi}{2}}$$

9. (Challenge) $f(x) = x^2$, $g(x) = 2x$ around x-axis.
Intersections: $x^2 = 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2$.
Test $x = 1$: $g(1) = 2 > f(1) = 1$. $R = 2x$, $r = x^2$.
$dV = \pi((2x)^2 - (x^2)^2)\,dx = \pi(4x^2 - x^4)\,dx$

$$V = \pi \int_0^2 (4x^2 - x^4)\,dx = \pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi \left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi \cdot \frac{2}{15} = \frac{64\pi}{15}$$

$$\boxed{V = \frac{64\pi}{15}}$$

10. (Challenge) $y = e^x$ around x-axis, $[0, 1]$.
$dV = \pi(e^x)^2\,dx = \pi e^{2x}\,dx$

$$V = \pi \int_0^1 e^{2x}\,dx = \pi \left[\frac{e^{2x}}{2}\right]_0^1 = \pi \left(\frac{e^2}{2} - \frac{1}{2}\right) = \frac{\pi(e^2 - 1)}{2}$$

$$\boxed{V = \frac{\pi(e^2 - 1)}{2}}$$

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End of Lesson • ApexMath
