Volumes of Revolution
Shell Method
ApexMath

Notes

A Different Way to Slice

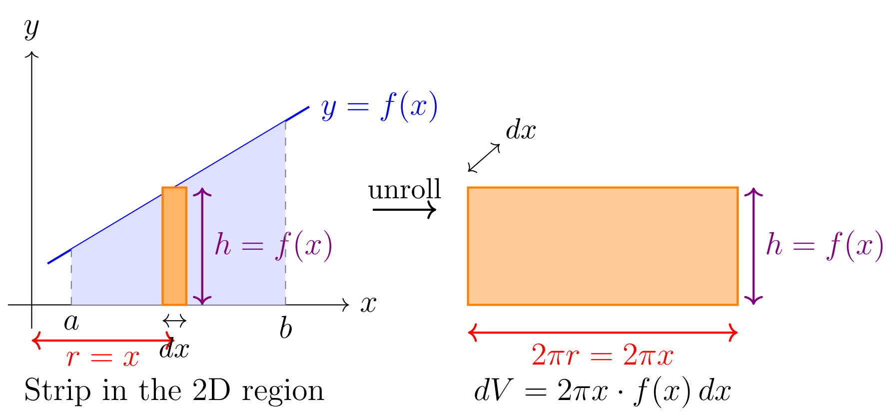
In the disk and washer method, we sliced the solid *perpendicular* to the axis of rotation. The shell method takes a completely different approach: we slice *parallel* to the axis, creating thin cylindrical shells like the layers of an onion.

Each shell is a thin hollow cylinder. If we were to cut it open and unroll it, it flattens out into a rectangular slab with:

- **Length** (circumference of the shell): $2\pi r$
- **Height**: h
- **Thickness**: dx (or dy)

So the volume of one thin shell is:

$$dV = 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$$



Shell Formula: Rotation Around the y-axis

When rotating around the y-axis, each vertical strip at position x forms a shell with:

- Radius $= x$
- Height $= f(x)$
- Thickness $= dx$

$$dV = 2\pi x\, f(x)\, dx \qquad \Longrightarrow \qquad \boxed{V = 2\pi \int_a^b x\, f(x)\, dx}$$

For the region between two curves $f(x) \geq g(x)$, the height of the shell is $f(x) - g(x)$:

$$dV = 2\pi x\, [f(x) - g(x)]\, dx \qquad \Longrightarrow \qquad V = 2\pi \int_a^b x\, [f(x) - g(x)]\, dx$$

Shell Formula: Rotation Around the x-axis

When rotating around the x-axis, use horizontal strips. The shell at height y has:

- Radius $= y$
- Height $= f(y)$ (the horizontal extent of the region at that y-value)
- Thickness $= dy$

$$dV = 2\pi y\, f(y)\, dy \qquad \Longrightarrow \qquad \boxed{V = 2\pi \int_c^d y\, f(y)\, dy}$$

Shells Around a Shifted Axis

When rotating around $x = k$ instead of $x = 0$, the radius of a shell at position x is the distance from x to the axis:

$$\text{radius} = |x - k|$$

For example, rotating around $x = -1$: each shell at position $x \geq 0$ has radius $x - (-1) = x + 1$.

$$dV = 2\pi(x + 1)\, f(x)\, dx$$

When to Use Shells vs. Washers

Situation	**Preferred Method**	**Why**
Rotating around y-axis, $y = f(x)$ given	Shells	Integrate directly in x
Rotating around x-axis, $y = f(x)$ given	Washers	Integrate directly in x
Region hard to express as $x = f(y)$	Shells	Avoids solving for x
Simple $x = f(y)$ form available	Washers	Integrate directly in y

Pro Tip

- Think of $dV = 2\pi \cdot r \cdot h \cdot \text{thickness}$ every time. Write all three pieces explicitly before integrating. Identifying the radius, height, and thickness correctly is the whole problem.
- The radius is always the distance from the shell to the axis of rotation. For the y-axis this is simply x. For a shifted axis $x = k$, it becomes $x - k$ (or $k - x$ if $x < k$).
- The height of the shell is the length of the vertical (or horizontal) strip, just like in area problems. For two curves it is top minus bottom (or right minus left).
- When in doubt about whether to use shells or washers, try both setups and pick whichever integral looks simpler to evaluate.
- Always double-check that the radius is positive across your entire interval of integration. If the axis passes through the region, you must split the integral.

Examples

Example 1 (Basic Shell Around y-axis): Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 2$ around the y-axis.

Step 1: Identify the pieces and write dV.
Rotating around the y-axis. Each vertical strip at position x forms a shell with radius $= x$ and height $= f(x) = x^2$.

$$dV = 2\pi \cdot x \cdot x^2 \, dx = 2\pi x^3 \, dx$$

Step 2: Integrate.

$$V = \int dV = 2\pi \int_0^2 x^3 \, dx = 2\pi \left[\frac{x^4}{4} \right]_0^2 = 2\pi \cdot 4 = 8\pi$$

$$\boxed{V = 8\pi}$$

Example 2 (Shell with Two Curves): Find the volume of the solid formed by rotating the region between $f(x) = \sqrt{x}$ and $g(x) = x^2$ from $x = 0$ to $x = 1$ around the y-axis.

Step 1: Identify the height of the shell.
Test $x = 0.5$: $f = \sqrt{0.5} \approx 0.707 > g = 0.25$. Height $= f(x) - g(x) = \sqrt{x} - x^2$.
Step 2: Write dV.

$$dV = 2\pi x \left(\sqrt{x} - x^2\right) dx = 2\pi(x^{3/2} - x^3) \, dx$$

Step 3: Integrate.

$$V = 2\pi \int_0^1 (x^{3/2} - x^3)\, dx = 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^4}{4} \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) = 2\pi \cdot \frac{3}{20}$$

$$\boxed{V = \frac{3\pi}{10}}$$

Example 3 (Shell Around Shifted Axis $x = -1$): Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 2$ around the line $x = -1$.

Step 1: Find the radius.

The axis is $x = -1$, which is to the left of the region. The radius of a shell at position $x \geq 0$ is:

$$r = x - (-1) = x + 1$$

Step 2: Write dV.

$$dV = 2\pi(x+1) \cdot x^2\, dx = 2\pi(x^3 + x^2)\, dx$$

Step 3: Integrate.

$$V = 2\pi \int_0^2 (x^3 + x^2)\, dx = 2\pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_0^2 = 2\pi \left(4 + \frac{8}{3} \right) = 2\pi \cdot \frac{20}{3} = \frac{40\pi}{3}$$

$$\boxed{V = \frac{40\pi}{3}}$$

Example 4 (Shell Around x-axis): Find the volume of the solid formed by rotating $x = y^2$ from $y = 0$ to $y = 2$ around the x-axis.

Step 1: Write dV using horizontal shells.

Each horizontal strip at height y forms a shell with radius $= y$ and height (horizontal extent) $= f(y) = y^2$.

$$dV = 2\pi y \cdot y^2\, dy = 2\pi y^3\, dy$$

Step 2: Integrate.

$$V = 2\pi \int_0^2 y^3\, dy = 2\pi \left[\frac{y^4}{4} \right]_0^2 = 2\pi \cdot 4 = 8\pi$$

$$\boxed{V = 8\pi}$$

Example 5 (Shells Easier Than Washers): Find the volume of the solid formed by rotating the region bounded by $y = x - x^2$ and $y = 0$ around the y-axis.

Note: Using the washer method here would require solving $y = x - x^2$ for x, giving $x = \frac{1 \pm \sqrt{1-4y}}{2}$ — two branches that need separate treatment. Shells are much cleaner.

Step 1: Find the limits. Zeros of $x - x^2 = x(1 - x)$: $x = 0$ and $x = 1$.

Step 2: Write dV.

$$dV = 2\pi x\,(x - x^2)\,dx = 2\pi(x^2 - x^3)\,dx$$

Step 3: Integrate.

$$V = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}$$

$$\boxed{V = \frac{\pi}{6}}$$

Common Mistakes

- **Forgetting the 2π factor:** The 2π comes from the circumference of the shell and is part of the formula from the start. It is not optional and cannot be added later.
- **Using the wrong radius for a shifted axis:** When rotating around $x = k$, the radius is $x - k$ (or $k - x$), not just x. Always write out the distance from the strip to the axis explicitly before building dV.
- **Getting the height wrong for two curves:** The height of the shell is top minus bottom (or right minus left for horizontal shells). Test a point to confirm which curve is which before writing the height.
- **Using shells when rotating around the x-axis but integrating in x:** When rotating around the x-axis with shells, you must use horizontal strips and integrate in y. Trying to set up shells in x while rotating around the x-axis gives the wrong formula.
- **Confusing the shell method with the disk method:** Disks/washers use πR^2; shells use $2\pi rh$. Mixing up the two formulas is a very common error when switching between methods.

Worksheet

1. Find the volume of the solid formed by rotating $y = x^3$ from $x = 0$ to $x = 2$ around the y-axis using shells.

2. Find the volume of the solid formed by rotating $y = 4 - x^2$ from $x = 0$ to $x = 2$ around the y-axis using shells.

3. Find the volume of the solid formed by rotating the region between $f(x) = x$ and $g(x) = x^2$ from $x = 0$ to $x = 1$ around the y-axis using shells.

4. Find the volume of the solid formed by rotating $y = e^x$ from $x = 0$ to $x = 1$ around the y-axis using shells.

5. Find the volume of the solid formed by rotating $y = \sin(x^2)$ from $x = 0$ to $x = \sqrt{\pi}$ around the y-axis.

   (Shells make this easy. A disk/washer approach would be very difficult.)

6. Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 3$ around the line $x = 4$ using shells.

   (Axis is to the right of the region. Radius = $4 - x$.)

7. Find the volume of the solid formed by rotating $y = \sqrt{x}$ from $x = 0$ to $x = 4$ around the line $x = -2$ using shells.

8. Find the volume of the solid formed by rotating the region bounded by $x = y^2$ and $x = 4$ around the x-axis using shells.

   (Horizontal shells. Radius = y, height = $4 - y^2$. Find y-limits first.)

9. (Challenge) Find the volume of the solid formed by rotating the region enclosed by $y = x^2$ and $y = 4x - x^2$ around the y-axis using shells.

   (Find intersection points first.)

10. (Challenge) The region bounded by $y = \ln(x)$, $y = 0$, and $x = e$ is rotated around the y-axis. Find the volume using shells.

Answer Key

1. $y = x^3$ around y-axis, $[0, 2]$.
$dV = 2\pi x \cdot x^3\,dx = 2\pi x^4\,dx$

$$V = 2\pi \int_0^2 x^4\,dx = 2\pi \left[\frac{x^5}{5}\right]_0^2 = 2\pi \cdot \frac{32}{5} = \frac{64\pi}{5}$$

$$\boxed{V = \frac{64\pi}{5}}$$

2. $y = 4 - x^2$ around y-axis, $[0, 2]$.
$dV = 2\pi x(4 - x^2)\,dx = 2\pi(4x - x^3)\,dx$

$$V = 2\pi \int_0^2 (4x - x^3)\,dx = 2\pi \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2\pi(8 - 4) = 8\pi$$

$$\boxed{V = 8\pi}$$

3. Region between $f(x) = x$ and $g(x) = x^2$, $[0, 1]$, around y-axis.
Test $x = 0.5$: $f = 0.5 > g = 0.25$. Height $= x - x^2$.
$dV = 2\pi x(x - x^2)\,dx = 2\pi(x^2 - x^3)\,dx$

$$V = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = 2\pi \cdot \frac{1}{12} = \frac{\pi}{6}$$

$$\boxed{V = \frac{\pi}{6}}$$

4. $y = e^x$ around y-axis, $[0, 1]$.
$dV = 2\pi x\, e^x\,dx$

$$V = 2\pi \int_0^1 xe^x\,dx$$

Use IBP: $u = x$, $dv = e^x\,dx$, $v = e^x$.

$$\int_0^1 xe^x\,dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx = e - \Big[e^x\Big]_0^1 = e - (e - 1) = 1$$

$$V = 2\pi(1) = 2\pi$$

$$\boxed{V = 2\pi}$$

5. $y = \sin(x^2)$ around y-axis, $[0, \sqrt{\pi}]$.
$dV = 2\pi x \sin(x^2)\,dx$

Use $u = x^2$, $du = 2x\,dx$:

$$V = 2\pi \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = 2\pi \cdot \frac{1}{2} \int_0^{\pi} \sin(u)\,du = \pi\Big[-\cos(u)\Big]_0^{\pi} = \pi(-\cos\pi + \cos 0) = \pi(1+1) = 2\pi$$

$$\boxed{V = 2\pi}$$

6. $y = x^2$ around $x = 4$, $[0, 3]$.
Axis is $x = 4$, to the right of region. Radius $= 4 - x$.
$dV = 2\pi(4 - x) \cdot x^2\,dx = 2\pi(4x^2 - x^3)\,dx$

$$V = 2\pi \int_0^3 (4x^2 - x^3)\,dx = 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^3 = 2\pi\left(36 - \frac{81}{4}\right) = 2\pi \cdot \frac{63}{4} = \frac{63\pi}{2}$$

$$\boxed{V = \frac{63\pi}{2}}$$

7. $y = \sqrt{x}$ around $x = -2$, $[0, 4]$.
Axis is $x = -2$, to the left of region. Radius $= x - (-2) = x + 2$.
$dV = 2\pi(x + 2)\sqrt{x}\,dx = 2\pi(x^{3/2} + 2x^{1/2})\,dx$

$$V = 2\pi \int_0^4 (x^{3/2} + 2x^{1/2})\,dx = 2\pi \left[\frac{2x^{5/2}}{5} + \frac{4x^{3/2}}{3}\right]_0^4$$

$$= 2\pi\left(\frac{2 \cdot 32}{5} + \frac{4 \cdot 8}{3}\right) = 2\pi\left(\frac{64}{5} + \frac{32}{3}\right) = 2\pi \cdot \frac{192 + 160}{15} = 2\pi \cdot \frac{352}{15} = \frac{704\pi}{15}$$

$$\boxed{V = \frac{704\pi}{15}}$$

8. $x = y^2$ and $x = 4$ around x-axis, horizontal shells.
y-limits: $y^2 = 4 \Rightarrow y = \pm 2$. By symmetry, integrate from 0 to 2 and double, or integrate from -2 to 2 noting that radius $= |y|$.
Using $y \in [0, 2]$ (top half) and doubling:
Each shell at height y: radius $= y$, height (horizontal extent) $= 4 - y^2$.
$dV = 2\pi y(4 - y^2)\,dy$

$$V = 2\pi \int_0^2 y(4 - y^2)\,dy \cdot 2$$

Wait — integrating from -2 to 2 handles both halves:

$$V = 2\pi \int_{-2}^{2} |y|(4 - y^2)\,dy = 2 \cdot 2\pi \int_0^2 y(4 - y^2)\,dy$$

$$= 4\pi \int_0^2 (4y - y^3)\,dy = 4\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 4\pi(8 - 4) = 16\pi$$

$$\boxed{V = 16\pi}$$

9. (Challenge) Region enclosed by $y = x^2$ and $y = 4x - x^2$ around y-axis.
Intersections: $x^2 = 4x - x^2 \Rightarrow 2x^2 - 4x = 0 \Rightarrow 2x(x-2) = 0 \Rightarrow x = 0, 2$.
Test $x = 1$: $f(1) = 4 - 1 = 3 > g(1) = 1$. Height $= (4x - x^2) - x^2 = 4x - 2x^2$.
$dV = 2\pi x(4x - 2x^2)\,dx = 2\pi(4x^2 - 2x^3)\,dx$

$$V = 2\pi \int_0^2 (4x^2 - 2x^3)\,dx = 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{2}\right]_0^2 = 2\pi\left(\frac{32}{3} - 8\right) = 2\pi \cdot \frac{8}{3} = \frac{16\pi}{3}$$

$$\boxed{V = \frac{16\pi}{3}}$$

10. (Challenge) Region bounded by $y = \ln(x)$, $y = 0$, $x = e$ around y-axis.

Shell at position x: radius $= x$, height $= \ln(x)$ (above $y = 0$), $x \in [1, e]$ (since $\ln(x) = 0$ at $x = 1$).

$dV = 2\pi x \ln(x)\,dx$

$$V = 2\pi \int_1^e x\ln(x)\,dx$$

Use IBP: $u = \ln(x)$, $dv = x\,dx$, $du = \frac{1}{x}\,dx$, $v = \frac{x^2}{2}$.

$$\int_1^e x\ln(x)\,dx = \left[\frac{x^2}{2}\ln(x)\right]_1^e - \int_1^e \frac{x}{2}\,dx = \frac{e^2}{2} - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{e^2 - 1}{4} = \frac{2e^2 - e^2 + 1}{4} = \frac{e^2 + 1}{4}$$

$$V = 2\pi \cdot \frac{e^2 + 1}{4} = \frac{\pi(e^2 + 1)}{2}$$

$$\boxed{V = \frac{\pi(e^2 + 1)}{2}}$$

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