Work
ApexMath

Notes

The Big Idea: When Force Is Not Constant

In physics, work is defined as force times distance:

$$W = F \cdot d$$

This formula works perfectly when the force is *constant* the entire time. For example, if you push a box with a steady $10\,\mathrm{N}$ force for $3\,\mathrm{m}$, then $W = 10 \times 3 = 30\,\mathrm{J}$.

But what if the force *changes* as you move? A stretched spring pushes back harder the more you stretch it. A cable gets lighter as you reel it in. Pumping water out of a tank requires more work to lift water from the bottom than from the top. In all of these cases, $W = F \cdot d$ no longer applies directly.

The solution is the same strategy used throughout this course: slice the problem into tiny pieces, compute a small amount of work dW for each piece (where the force is approximately constant over that tiny distance), then integrate to get the total.

$$dW = F(x)\,dx \qquad \Longrightarrow \qquad \boxed{W = \int_a^b F(x)\,dx}$$

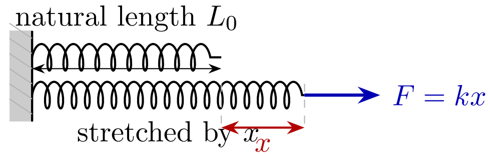
Here $F(x)$ is the force at position x, and the integral adds up all the tiny contributions from $x = a$ to $x = b$.

Application 1: Springs and Hooke's Law

A spring has a natural resting length. When you stretch or compress it, it resists. **Hooke's Law** says the force required to hold a spring stretched (or compressed) by a distance x from its natural length is:

$$F(x) = kx$$

where $k > 0$ is the **spring constant** (measured in N/m or lb/ft). The farther you stretch it, the harder you have to pull.



The work to stretch a spring from $x = a$ to $x = b$ (where x measures displacement from the natural length) is:

$$W = \int_a^b kx\,dx = k\left[\frac{x^2}{2}\right]_a^b = \frac{k}{2}(b^2 - a^2)$$

Application 2: Lifting Problems (Cables and Chains)

When you lift a heavy cable or chain by winding it onto a spool, the amount of cable still hanging decreases as you go. Less hanging cable means less weight to lift as you reel it in.

Set $x = 0$ at the top (the spool) and $x = L$ at the bottom. At position x, the remaining hanging length is $(L - x)$, so if the cable has weight density ρ (weight per unit length), the force needed at position x is:

$$F(x) = \rho(L - x)$$

The work to reel in the entire cable from $x = 0$ to $x = L$ is:

$$W = \int_0^L \rho(L - x)\,dx$$

Application 3: Pumping Liquids

To pump liquid out of a tank, you must lift every small layer of liquid up to the outlet at the top (or to some height above). The deeper a layer sits, the farther it must travel, so it requires more work.

The key setup is:

- Let x measure depth from the top of the tank downward, so $x = 0$ is at the top and $x = h$ is at the bottom.
- A thin layer of liquid at depth x has thickness dx.
- Its volume is $A(x)\,dx$, where $A(x)$ is the cross-sectional area of the tank at depth x.
- Its weight is $\rho \cdot A(x)\,dx$, where ρ is the weight density of the liquid (e.g. water $\approx$ $62.4\,\mathrm{lb/ft^3}$ or $9800\,\mathrm{N/m^3}$).
- It must be lifted a distance of x to reach the top.
- So $dW = \rho \cdot x \cdot A(x)\,dx$.

$$\boxed{W = \int_0^h \rho \cdot x \cdot A(x)\,dx}$$

If the outlet is *above* the top of the tank by a distance d, then each layer at depth x must travel $x + d$, and the formula becomes:

$$W = \int_0^h \rho(x + d)\,A(x)\,dx$$

Pro Tip

- **Always define your coordinate system first.** Before writing any formula, decide where $x = 0$ is and which direction x increases. Errors in pumping problems almost always trace back to an unclear coordinate setup.

- **For springs, x measures displacement from the natural length, not from the wall.** The natural length of the spring is irrelevant; only how far you stretch or compress it from that rest position matters.

- **For pumping problems, the distance a layer travels equals its depth below the outlet.** If the outlet is at the very top of the tank and x is measured downward from the top, the distance is just x. Draw a picture and label the distances before setting up the integral.

- **Check your units.** Work is in joules (J) when force is in newtons (N) and distance is in meters (m), or in foot-pounds (ft·lb) when using the imperial system. Make sure ρ, k, and all lengths use consistent units.

- **For cables, decide whether the problem gives you weight density (weight per unit length) or total weight.** If given total weight W_0 and length L, then $\rho = W_0/L$.

Examples

Example 1 (Spring — Finding Work from Stretch): A spring has spring constant $k = 40\,\text{N/m}$. Find the work required to stretch it from its natural length out to $0.3\,\text{m}$.

Step 1: Set up the integral.

The spring starts at its natural length, so $x = 0$. It is stretched to $x = 0.3\,\text{m}$. By Hooke's Law, $F(x) = kx = 40x$.

$$W = \int_0^{0.3} 40x\,dx$$

Step 2: Integrate.

$$W = 40\left[\frac{x^2}{2}\right]_0^{0.3} = 20\left[x^2\right]_0^{0.3} = 20(0.09) - 20(0)$$

$$\boxed{W = 1.8\text{ J}}$$

Example 2 (Spring — Finding k First): A spring requires $6\,\text{J}$ of work to stretch it $0.5\,\text{m}$ beyond its natural length. How much work is required to stretch it an additional $0.3\,\text{m}$ (i.e., from $0.5\,\text{m}$ to $0.8\,\text{m}$)?

Step 1: Find the spring constant k.

We know the work to stretch from 0 to $0.5\,\text{m}$ is $6\,\text{J}$:

$$6 = \int_0^{0.5} kx\,dx = k\left[\frac{x^2}{2}\right]_0^{0.5} = k \cdot \frac{0.25}{2} = \frac{k}{8}$$

Solving for k:

$$k = 6 \times 8 = 48 \text{ N/m}$$

Step 2: Set up the integral for the additional stretch.
Now integrate from $x = 0.5$ to $x = 0.8$ using $F(x) = 48x$:

$$W = \int_{0.5}^{0.8} 48x\,dx = 48\left[\frac{x^2}{2}\right]_{0.5}^{0.8} = 24\left[x^2\right]_{0.5}^{0.8}$$

Step 3: Evaluate.

$$W = 24[(0.8)^2 - (0.5)^2] = 24[0.64 - 0.25] = 24(0.39)$$

$$\boxed{W = 9.36 \text{ J}}$$

Example 3 (Cable): A 10 m cable weighing 4 N/m hangs vertically from a winch. Find the work required to reel in the entire cable.

Step 1: Set up the coordinate system.
Let x = distance already reeled in, measured from the top downward. At position x, the remaining hanging length is $(10 - x)$.
The force at position x is the weight of the remaining cable:

$$F(x) = 4(10 - x)$$

Step 2: Set up the integral.
The cable is reeled in from $x = 0$ (nothing reeled in yet) to $x = 10$ (fully reeled in):

$$W = \int_0^{10} 4(10 - x)\,dx$$

Step 3: Integrate.

$$W = 4\int_0^{10} (10 - x)\,dx = 4\left[10x - \frac{x^2}{2}\right]_0^{10} = 4\,[100 - 50] = 4(50)$$

$$\boxed{W = 200 \text{ J}}$$

Example 4 (Pumping Water — Rectangular Tank): A rectangular tank is 4 ft long, 3 ft wide, and 5 ft deep. It is full of water (weight density 62.4 lb/ft^3). Find the work required to pump all the water out over the top edge.

Step 1: Set up the coordinate system.

Let x = depth measured downward from the top of the tank. So $x = 0$ is the top (where water exits) and $x = 5$ is the bottom.

Step 2: Find the volume and weight of a thin layer at depth x.

The cross-sectional area is constant: $A = 4 \times 3 = 12\,\text{ft}^2$.

Volume of thin layer: $dV = 12\,dx$

Weight of thin layer: $dF = 62.4 \times 12\,dx = 748.8\,dx$

Step 3: Find the distance the layer must travel.

A layer at depth x must be lifted x feet to reach the top:

$$dW = 748.8\,x\,dx$$

Step 4: Integrate from top to bottom.

$$W = \int_0^5 748.8\,x\,dx = 748.8\left[\frac{x^2}{2}\right]_0^5 = 748.8 \cdot \frac{25}{2} = 748.8 \cdot 12.5$$

$$\boxed{W = 9,360 \text{ ft}\cdot\text{lb}}$$

Example 5 (Pumping Water — Conical Tank): A conical tank (point down) has height 10 m and radius 4 m at the top. It is full of water ($\rho = 9800\,\text{N/m}^3$). Find the work to pump all water out over the top.

Step 1: Set up the coordinate system.

Let x = depth measured downward from the top, so $x = 0$ is the top and $x = 10$ is the bottom (the tip of the cone).

Step 2: Find $A(x)$, the cross-sectional area at depth x.

By similar triangles, the radius at depth x from the top (i.e., at height $10 - x$ from the bottom tip) is:

$$r(x) = \frac{4}{10}(10 - x) = \frac{2(10 - x)}{5}$$

The cross-sectional area is:

$$A(x) = \pi[r(x)]^2 = \pi\left(\frac{2(10 - x)}{5}\right)^2 = \frac{4\pi(10 - x)^2}{25}$$

Step 3: Write dW.

Each thin layer at depth x must be lifted x meters:

$$dW = 9800 \cdot x \cdot \frac{4\pi(10 - x)^2}{25}\,dx = \frac{39200\pi}{25}\,x(10 - x)^2\,dx$$

Step 4: Expand $x(10 - x)^2$.

$$x(10 - x)^2 = x(100 - 20x + x^2) = 100x - 20x^2 + x^3$$

Step 5: Integrate.

$$W = \frac{39200\pi}{25}\int_0^{10}(100x - 20x^2 + x^3)\,dx = \frac{39200\pi}{25}\left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4}\right]_0^{10}$$

At $x = 10$:

$$50(100) - \frac{20(1000)}{3} + \frac{10000}{4} = 5000 - \frac{20000}{3} + 2500 = 7500 - \frac{20000}{3} = \frac{22500 - 20000}{3} = \frac{2500}{3}$$

Therefore:

$$W = \frac{39200\pi}{25}\cdot\frac{2500}{3} = \frac{39200\pi \times 100}{3} = \frac{3,920,000\pi}{3}$$

$$\boxed{W = \frac{3,920,000\pi}{3} \approx 4,105,284 \text{ J}}$$

Common Mistakes

- **Using $W = F \cdot d$ when the force is variable:** This formula only works for constant force. As soon as the force changes with position, you must set up an integral.
- **Forgetting to find k before setting up the spring integral:** If the problem gives you a known work or force at a specific displacement, use that information to solve for k first, then proceed.
- **Measuring the wrong distance in pumping problems:** The distance a layer of liquid travels is measured from the layer to the *outlet*, not to the bottom of the tank. Always draw and label the setup before writing dW.
- **Using the wrong cross-sectional area for non-rectangular tanks:** For cylinders, $A(x)$ is constant. For cones and other shapes, $A(x)$ changes with depth. Use similar triangles or the geometry of the tank to find $r(x)$ first.
- **Inconsistent units:** Make sure all quantities — length, weight density, spring constant — are in the same unit system throughout the problem.

Worksheet

1. A spring has spring constant $k = 30\,\text{N/m}$. Find the work required to stretch it $0.4\,\text{m}$ from its natural length.

2. A spring requires 10 J of work to stretch it 0.5 m beyond its natural length. Find the spring constant k.

3. Using the spring from problem 2, find the work to stretch it from 0.5 m to 1 m beyond its natural length.

4. A 20 ft chain weighing 3 lb/ft hangs vertically from a ledge. Find the work required to pull the entire chain up to the ledge.

5. A 15 m cable weighing 5 N/m hangs from a winch. Find the work to reel in the top half of the cable (from $x = 0$ to $x = 7.5\,\mathrm{m}$).

6. A rectangular tank is 5 m long, 2 m wide, and 4 m deep. It is full of water ($\rho = 9800\,\mathrm{N/m^3}$). Find the work to pump all water out over the top.

7. A cylindrical tank has radius 3 ft and height 8 ft. It is half full of water ($62.4\,\mathrm{lb/ft^3}$). Find the work to pump the water out over the top edge.

8. (Challenge) A conical tank (point down) has height $6\,\mathrm{m}$ and top radius $2\,\mathrm{m}$. It is full of water ($9800\,\mathrm{N/m^3}$). Find the work to pump all water out to an outlet $1\,\mathrm{m}$ above the top of the tank.

Answer Key

1. Spring with $k = 30\,\text{N/m}$, stretched $0.4\,\text{m}$ from natural length.

Step 1: Write $F(x) = kx = 30x$.
Step 2: Set up the integral from $x = 0$ to $x = 0.4$.

$$W = \int_0^{0.4} 30x\,dx = 30\left[\frac{x^2}{2}\right]_0^{0.4} = 15\left[x^2\right]_0^{0.4}$$

Step 3: Evaluate.

$$W = 15(0.4)^2 - 15(0)^2 = 15(0.16)$$

$$\boxed{W = 2.4\text{ J}}$$

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2. Spring requires $10\,\text{J}$ to stretch $0.5\,\text{m}$. Find k.

Step 1: Write the work integral in terms of k.

$$W = \int_0^{0.5} kx\,dx = k\left[\frac{x^2}{2}\right]_0^{0.5} = k \cdot \frac{0.25}{2} = \frac{k}{8}$$

Step 2: Set equal to 10 and solve.

$$\frac{k}{8} = 10 \implies k = 80$$

$$\boxed{k = 80\text{ N/m}}$$

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3. Spring with $k = 80\,\text{N/m}$ (from problem 2). Work from $0.5\,\text{m}$ to $1\,\text{m}$.

Step 1: Write $F(x) = 80x$.
Step 2: Set up the integral from $x = 0.5$ to $x = 1$.

$$W = \int_{0.5}^{1} 80x\,dx = 80\left[\frac{x^2}{2}\right]_{0.5}^{1} = 40\left[x^2\right]_{0.5}^{1}$$

Step 3: Evaluate.

$$W = 40[(1)^2 - (0.5)^2] = 40[1 - 0.25] = 40(0.75)$$

$$\boxed{W = 30\text{ J}}$$

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4. Chain: $L = 20\,\text{ft}$, $\rho = 3\,\text{lb/ft}$.

Step 1: Set x = distance pulled up from the starting position (0 to 20 ft).
At position x, the remaining hanging length is $(20 - x)$ ft.

Step 2: Write $F(x)$.

$$F(x) = 3(20 - x)$$

Step 3: Integrate from $x = 0$ to $x = 20$.

$$W = \int_0^{20} 3(20 - x)\,dx = 3\left[20x - \frac{x^2}{2}\right]_0^{20}$$

Step 4: Evaluate at $x = 20$.

$$W = 3\,[400 - 200] = 3(200)$$

$$\boxed{W = 600 \text{ ft}\cdot\text{lb}}$$

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5. Cable: $L = 15\,\text{m}$, $\rho = 5\,\text{N/m}$. Reel in top half ($x = 0$ to $7.5\,\text{m}$).

Step 1: At position x, remaining hanging cable has length $(15 - x)$.

$$F(x) = 5(15 - x)$$

Step 2: Integrate from $x = 0$ to $x = 7.5$.

$$W = \int_0^{7.5} 5(15 - x)\,dx = 5\left[15x - \frac{x^2}{2}\right]_0^{7.5}$$

Step 3: Evaluate at $x = 7.5$.

$$15(7.5) - \frac{(7.5)^2}{2} = 112.5 - \frac{56.25}{2} = 112.5 - 28.125 = 84.375$$

$$W = 5(84.375)$$

$$\boxed{W = 421.875 \text{ J}}$$

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6. Rectangular tank: $5\,\text{m} \times 2\,\text{m} \times 4\,\text{m}$ deep, full of water, $\rho = 9800\,\text{N/m}^3$.

Step 1: Let x = depth from the top. Outlet is at $x = 0$, bottom is at $x = 4$.
Step 2: Cross-sectional area is constant: $A = 5 \times 2 = 10\,\text{m}^2$.
Step 3: Write dW.

$$dW = 9800 \cdot x \cdot 10\,dx = 98000\,x\,dx$$

Step 4: Integrate from $x = 0$ to $x = 4$.

$$W = \int_0^4 98000\,x\,dx = 98000\left[\frac{x^2}{2}\right]_0^4 = 49000\left[x^2\right]_0^4$$

Step 5: Evaluate.

$$W = 49000(16)$$

$$\boxed{W = 784,000 \text{ J}}$$

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7. Cylindrical tank: radius 3 ft, height 8 ft, half full of water ($62.4\,\text{lb/ft}^3$).

Step 1: Let x = depth from the top of the tank. The tank is half full, so the water fills from $x = 4$ (the water surface) to $x = 8$ (the bottom).

The outlet is at the top, $x = 0$. Each layer at depth x must be lifted x feet.

Step 2: Cross-sectional area is constant: $A = \pi r^2 = 9\pi\,\text{ft}^2$.

Step 3: Write dW.

$$dW = 62.4 \cdot x \cdot 9\pi\,dx = 561.6\pi\,x\,dx$$

Step 4: Integrate from $x = 4$ to $x = 8$ (where the water is).

$$W = \int_4^8 561.6\pi\,x\,dx = 561.6\pi\left[\frac{x^2}{2}\right]_4^8 = 280.8\pi\left[x^2\right]_4^8$$

Step 5: Evaluate.

$$W = 280.8\pi\left[(8)^2 - (4)^2\right] = 280.8\pi[64 - 16] = 280.8\pi(48)$$

$$\boxed{W = 13,478.4\pi \approx 42,336 \text{ ft·lb}}$$

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8. (Challenge) Conical tank (point down): height 6 m, top radius 2 m, full of water ($9800\,\text{N/m}^3$), outlet 1 m above the top.

Step 1: Set x = depth measured downward from the top of the tank. So $x = 0$ is the rim and $x = 6$ is the tip at the bottom. The outlet is 1 m above the rim, so each layer at depth x must travel $x + 1$ meters total.

Step 2: Find $r(x)$ using similar triangles.

At depth x from the top, the distance from the tip (bottom) is $(6 - x)$. The radius at the top is 2 when the full height is 6, so:

$$r(x) = \frac{2}{6}(6 - x) = \frac{6 - x}{3}$$

Step 3: Find $A(x)$.

$$A(x) = \pi[r(x)]^2 = \pi\left(\frac{6 - x}{3}\right)^2 = \frac{\pi(6 - x)^2}{9}$$

Step 4: Write dW.

Each layer travels $(x + 1)$ meters:

$$dW = 9800(x+1) \cdot \frac{\pi(6-x)^2}{9}\,dx = \frac{9800\pi}{9}(x+1)(6-x)^2\,dx$$

Step 5: Expand $(x+1)(6-x)^2$.
First expand $(6-x)^2 = 36 - 12x + x^2$. Then multiply by $(x+1)$:

$$(x+1)(36-12x+x^2) = 36x - 12x^2 + x^3 + 36 - 12x + x^2 = x^3 - 11x^2 + 24x + 36$$

Step 6: Integrate from $x = 0$ to $x = 6$.

$$W = \frac{9800\pi}{9}\int_0^6 (x^3 - 11x^2 + 24x + 36)\,dx = \frac{9800\pi}{9}\left[\frac{x^4}{4} - \frac{11x^3}{3} + 12x^2 + 36x\right]_0^6$$

At $x = 6$:

$$\frac{1296}{4} - \frac{11(216)}{3} + 12(36) + 36(6) = 324 - 792 + 432 + 216 = 180$$

$$W = \frac{9800\pi}{9} \cdot 180 = 9800\pi \cdot 20$$

$$\boxed{W = 196,000\pi \approx 615,752 \text{ J}}$$

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