# Functions & Relations Mastery Pack: Exponential Functions & Euler's Number $e$

Tutoring Notes

## Lesson: Exponential Functions & Euler's Number $e$

### Notes: Exponential Functions

**Definition:** An exponential function is a function where the variable appears in the exponent:

$$f(x) = a \cdot b^x$$

where:

- $a \neq 0$ is the initial value
- $b > 0$, $b \neq 1$ is the base
- $x$ is the variable

**Properties:**

- Domain: all real numbers $(-\infty, \infty)$
- Range: $y > 0$ if $a > 0$, $y < 0$ if $a < 0$
- Horizontal asymptote: $y = 0$
- Growth/decay depends on $b$:
  - $b > 1$: exponential growth
  - $0 < b < 1$: exponential decay

**Tip:** When something grows, it keeps getting bigger by the same amount each time. When something decays, it keeps getting smaller by the same amount each time.

## Euler's Number $e$

Euler's number, denoted as $e \approx 2.718$, is a special number that naturally arises when growth happens continuously.

One way to understand $e$ is by the function:

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

**Explanation:** This is a common function measuring growth, and as $x$ becomes very large, the value of $f(x)$ gets closer and closer to 2.718, which we call $e$. This helps us understand continuous growth, like continuously compounded interest or natural population growth.

## Walkthrough Examples

**Example 1: Exponential Growth**

A population of 5000 grows 4% per year. Find the population after 5 years.

Step 1: Formula:

$$P = P_0(1 + r)^t$$

Step 2: Substitute values:

$$P = 5000(1 + 0.04)^5 = 5000(1.04)^5$$

Step 3: Calculate:

$$1.04^5 \approx 1.21665$$

$$P \approx 5000 \cdot 1.21665 = 6083$$

Answer: $\boxed{6083 \text{ people}}$

**Example 2: Exponential Decay**

Radioactive substance: 80 g, half-life 3 hours, after 6 hours.

Step 1: Formula:

$$N = N_0 \left(\frac{1}{2}\right)^{t/h}$$

Step 2: Substitute:

$$N = 80 \left(\frac{1}{2}\right)^{6/3} = 80 \left(\frac{1}{2}\right)^2$$

Step 3: Calculate:

$$N = 80 \cdot \frac{1}{4} = 20$$

Answer: $\boxed{20 \text{ grams}}$

**Example 3: Continuous Growth Using $e$**

Deposit \$1000 in a bank account earning 5% interest, compounded continuously, after 3 years.

Step 1: Formula for continuous growth:

$$A = Pe^{rt}$$

Step 2: Substitute values:

$$A = 1000 \cdot e^{0.05 \cdot 3} = 1000 \cdot e^{0.15}$$

Step 3: Approximate:

$$e^{0.15} \approx 1.1618$$

$$A \approx 1000 \cdot 1.1618 = 1161.80$$

Answer: $\boxed{\$1161.80}$

## Worksheet

**Problem 1:** A car is worth \$25,000 and decreases 15% per year. Find its value after 4 years.

**Problem 2:** Bacteria: 200, doubles every 3 hours. Find the amount after 9 hours.

**Problem 3:** A town: 10,000 residents, grows 2% per year. Find the population after 10 years.

**Problem 4:** Substance: 160 g, half-life 8 days, after 24 days.

**Problem 5:** Solve for $x$: $2^x = 32$

**Problem 6:** Solve for $x$: $5^{2x} = 125$

**Problem 7:** (Derive formula) Population 50,000, grows 3% per year. Derive formula and find after 10 years.

**Problem 8:** (Derive formula) Deposit \$1000, 5% annual interest. Derive formula for $n$ years and find after 7 years.

**Problem 9:** (Continuous growth) \$2000 at 4% interest, compounded continuously, after 5 years.

**Problem 10:** (Continuous growth) 150 rabbits, grows 15% per year continuously, after 3 years.

## Answer Key

**Problem 1:** $V = 25000(1 - 0.15)^4 = 25000(0.85)^4 \approx 13050$ Answer: $\boxed{\$13{,}050}$

**Problem 2:** $N = 200 \cdot 2^{9/3} = 200 \cdot 2^3 = 200 \cdot 8 = 1600$ Answer: $\boxed{1600 \text{ bacteria}}$

**Problem 3:** $P = 10000(1.02)^{10} \approx 12190$ Answer: $\boxed{12{,}190 \text{ residents}}$

**Problem 4:** $N = 160(1/2)^{24/8} = 160 \cdot (1/2)^3 = 20$ Answer: $\boxed{20 \text{ grams}}$

**Problem 5:** $2^x = 32 \implies x = 5$ Answer: $\boxed{5}$

**Problem 6:** $5^{2x} = 125 = 5^3 \implies 2x = 3 \implies x = 3/2$ Answer: $\boxed{3/2}$

**Problem 7:** Formula: $P = 50000(1.03)^t$ After 10 years: $P \approx 67156$ Answer: $\boxed{67{,}156 \text{ people}}$

**Problem 8:** Formula: $A = 1000(1.05)^n$ After 7 years: $A \approx 1407$ Answer: $\boxed{\$1{,}407}$

**Problem 9:** $A = 2000e^{0.04 \cdot 5} \approx 2000 \cdot 1.2214 = 2442.80$ Answer: $\boxed{\$2{,}442.80}$

**Problem 10:** $P = 150e^{0.15 \cdot 3} \approx 150 \cdot 1.5683 = 235.25$ Answer: $\boxed{\text{Approximately 235 rabbits}}$

# Thank You!

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