Functions & Relations Mastery Pack: Review of Exponential & Logarithmic Functions

ApexMath

Lesson: Comprehensive Review

Notes

This review combines all skills you've practiced:

1. **Exponential Functions:** Identify growth/decay, write $y = ab^t$ or $y = Ae^{kt}$.
2. **Logarithmic Functions:** Recall $\log_b(x)$ is the inverse of b^x.
3. **Logarithm Properties:** Use $\log_b(MN) = \log_b M + \log_b N$, $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$, $\log_b(M^k) = k \log_b M$.
4. **Solving Equations:** Reduce exponentials/logs into solvable forms by applying inverse operations.
5. **Applications:** Build equations from words (half-life, population, cooling, scales) and solve in context.

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Deep Walkthrough Examples

Example A. Exponential vs Logarithm
Solve for x: $5^{2x-1} = 200$

Step 1. Take log:

$$(2x - 1)\log 5 = \log 200$$

Step 2. Isolate:

$$2x - 1 = \frac{\log 200}{\log 5}$$

Step 3. Solve:

$$2x = \frac{\log 200}{\log 5} + 1$$

$$x = \frac{1}{2}\left(\frac{\log 200}{\log 5} + 1\right) \approx 2.321$$

Final Answer: $\boxed{2.321}$

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Example B. Using Log Properties
Simplify: $\log_2\left(\frac{x^4y^3}{\sqrt{z}}\right)$

Step 1. Break fraction:

$$\log_2(x^4y^3) - \log_2(\sqrt{z})$$

Step 2. Expand numerator:

$$\log_2(x^4) + \log_2(y^3) - \tfrac{1}{2}\log_2(z)$$

Step 3. Apply power rule:

$$4\log_2 x + 3\log_2 y - \tfrac{1}{2}\log_2 z$$

Final Answer: $\boxed{4\log_2 x + 3\log_2 y - \tfrac{1}{2}\log_2 z}$

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Example C. Application Problem
A bacteria culture doubles every 6 hours. Starting with 500, how long until population = 40,000?

Step 1. Model:

$$P(t) = 500 \cdot 2^{t/6}$$

Step 2. Target:

$$40000 = 500 \cdot 2^{t/6} \quad \Rightarrow \quad 80 = 2^{t/6}$$

Step 3. Take log:

$$\frac{t}{6} = \log_2(80) = \frac{\ln 80}{\ln 2}$$

Step 4. Solve:

$$t = 6 \cdot \frac{\ln 80}{\ln 2} \approx 35.822$$

Final Answer: $\boxed{35.822 \text{ hours}}$

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Common Mistakes to Avoid

- Forgetting to take log of both sides when isolating an exponent.
- Misusing log laws (e.g., $\log(M+N) \neq \log M + \log N$).
- Dropping units in application problems.
- Rounding too early; only round final answers.
- Forgetting domain restrictions: arguments of logs must be positive.

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Worksheet Problems

1. Solve for x: $3^{2x+1} = 500$.

2. Expand and simplify: $\log_5\left(\dfrac{25a^3}{\sqrt[3]{b}}\right)$.

3. Radioactive isotope half-life = 30 years. Initial mass 200 g. After how long will only 25 g remain?

4. Solve for x: $\ln(x-1) + \ln(x+1) = \ln(20)$.

5. A population grows continuously at 12% per year. Initial = 2000. In how many years will it reach 20,000?

6. Earthquake magnitudes: 7.8 and 6.3. How many times stronger is the first?

7. A cup of tea cools from 95°C to 60°C in 10 minutes in a 20°C room. When will it cool to 30°C?

8. Solve for x: $4^{x-2} = 7$.

9. Simplify fully: $\log\left(\dfrac{x^2y^5}{\sqrt{z^3}}\right)$.

10. A country's GDP follows $P(t) = P_0e^{kt}$. If $P(2000) = 400$B and $P(2010) = 800$B, predict GDP in 2030.

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Detailed Answer Key

1. $3^{2x+1} = 500$

$$(2x+1)\ln 3 = \ln 500$$

$$2x+1 = \frac{\ln 500}{\ln 3}$$

$$x = \frac{1}{2}\left(\frac{\ln 500}{\ln 3} - 1\right) \approx 3.595$$

Final Answer: $\boxed{3.595}$

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2. $\log_5\left(\frac{25a^3}{\sqrt[3]{b}}\right)$

$$= \log_5 25 + \log_5 a^3 - \log_5(b^{1/3})$$

$$= 2 + 3\log_5 a - \tfrac{1}{3}\log_5 b$$

Final Answer: $\boxed{2 + 3\log_5 a - \frac{1}{3}\log_5 b}$

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3. Half-life problem

$$200\left(\tfrac{1}{2}\right)^{t/30} = 25$$

$$\left(\tfrac{1}{2}\right)^{t/30} = \tfrac{1}{8}$$

$$\frac{t}{30} = 3 \quad \Rightarrow \quad t = 90$$

Final Answer: $\boxed{\text{90 years}}$

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4. $\ln(x-1) + \ln(x+1) = \ln(20)$

$$\ln[(x-1)(x+1)] = \ln 20$$

$$\ln(x^2-1) = \ln 20$$

$$x^2 - 1 = 20 \quad \Rightarrow \quad x^2 = 21$$

$$x = \pm\sqrt{21}$$

Domain check: $x - 1 > 0 \Rightarrow x > 1$ So $x = \sqrt{21} \approx 4.583$

Final Answer: $\boxed{4.583}$

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5. Population

$$2000e^{0.12t} = 20000$$
$$e^{0.12t} = 10$$
$$t = \frac{\ln 10}{0.12} \approx 19.188$$

Final Answer: $\boxed{19.188 \text{ years}}$

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6. Earthquakes

$$7.8 - 6.3 = 1.5 = \log_{10}\left(\tfrac{I_1}{I_2}\right)$$
$$\tfrac{I_1}{I_2} = 10^{1.5} \approx 31.623$$

Final Answer: $\boxed{31.623 \text{ times}}$

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7. Cooling Model: $T(t) = 20 + (95 - 20)e^{kt} = 20 + 75e^{kt}$

At $t = 10$, $T = 60$:

$$60 = 20 + 75e^{10k} \quad \Rightarrow \quad e^{10k} = \tfrac{40}{75} = \tfrac{8}{15}$$
$$k = \tfrac{1}{10}\ln\left(\tfrac{8}{15}\right) \approx -0.0580$$

Now for $T = 30$:

$$30 = 20 + 75e^{kt} \quad \Rightarrow \quad e^{kt} = \tfrac{10}{75} = \tfrac{2}{15}$$
$$t = \frac{\ln(2/15)}{k} \approx 31.469$$

Final Answer: $\boxed{31.469 \text{ minutes}}$

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8. $4^{x-2} = 7$

$$(x-2)\ln 4 = \ln 7$$
$$x - 2 = \frac{\ln 7}{\ln 4}$$
$$x = 2 + \frac{\ln 7}{\ln 4} \approx 3.404$$

Final Answer: $\boxed{3.404}$

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9. $\log\left(\dfrac{x^2y^5}{\sqrt{z^3}}\right)$

$$= \log x^2 + \log y^5 - \log(z^{3/2})$$

$$= 2\log x + 5\log y - \tfrac{3}{2}\log z$$

Final Answer: $\boxed{2\log x + 5\log y - \tfrac{3}{2}\log z}$

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10. GDP Model $P(t) = 400e^{kt}$, with $t = 0$ at 2000.

At $t = 10$, $P = 800$:

$$800 = 400e^{10k} \quad \Rightarrow \quad e^{10k} = 2$$

$$k = \tfrac{\ln 2}{10} \approx 0.0693$$

At 2030, $t = 30$:

$$P(30) = 400e^{30k} = 400e^{30(0.0693)}$$

$$= 400e^{2.079} \approx 400 \cdot 7.999 \approx 3199.6$$

Final Answer: $\boxed{\$3.200 \text{ trillion}}$

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