Lesson 5: Factoring Higher-Degree Polynomials

ApexMath

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Notes

In this lesson, we learn how to factor higher-degree polynomials using the **Factor Theorem**, the **Remainder Theorem**, and **synthetic division**.

Factor Theorem

The **Factor Theorem** states: If $P(c) = 0$, then $(x - c)$ is a factor of $P(x)$.

Remainder Theorem

The **Remainder Theorem** states: If $P(x)$ is divided by $(x - c)$, then the remainder is $P(c)$. If $P(c) = 0$, then $(x - c)$ is a factor.

Rational Root Theorem

The **Rational Root Theorem** helps us guess possible rational roots of a polynomial. For a polynomial with integer coefficients:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

Possible rational roots are:

$$\pm \frac{\text{factors of constant term } (a_0)}{\text{factors of leading coefficient } (a_n)}.$$

Synthetic Division (Step by Step)

Synthetic division is a shortcut for dividing polynomials by linear factors $(x - c)$.

Example table setup for dividing $P(x)$ by $(x - c)$:

$$\begin{array}{c|cccc} c & a_n & a_{n-1} & \cdots & a_0 \\ \hline & & & & \end{array}$$

Step by step: 1. Bring the first coefficient straight down. 2. Multiply by c, write result below next coefficient. 3. Add down the column. 4. Repeat until the end. The last number is the remainder.

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Walkthrough Example 1 (Given Factor)

Factor $P(x) = 2x^3 - 3x^2 - 11x + 6$ given that $(x - 3)$ is a factor.

Step 1: Setup synthetic division

We test with $c = 3$ (because $(x-3)$ means $c = 3$). Coefficients: $[2, -3, -11, 6]$

$$\begin{array}{c|cccc} 3 & 2 & -3 & -11 & 6 \\ \hline & & & & \end{array}$$

Step 2: Bring down first coefficient

Bring down 2:

$$\begin{array}{c|cccc} 3 & 2 & -3 & -11 & 6 \\ \hline & & & & \\ & 2 & & & \end{array}$$

Step 3: Multiply and add step by step

- Multiply $2 \cdot 3 = 6$, add to -3: 3. - Multiply $3 \cdot 3 = 9$, add to -11: -2. - Multiply $-2 \cdot 3 = -6$, add to 6: 0 (remainder).

$$
\begin{array}{r|rrrr}
3 & 2 & -3 & -11 & 6 \\
\hline
 & & 6 & 9 & -6 \\
 & 2 & 3 & -2 & 0
\end{array}
$$

Quotient: $2x^2 + 3x - 2$. Factor that quadratic: $2x^2 + 3x - 2 = (2x - 1)(x + 2)$.

Final Answer: $P(x) = (x - 3)(2x - 1)(x + 2)$

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Walkthrough Example 2 (No Factor Given)

Factor $Q(x) = x^3 - 6x^2 + 11x - 6$.

Step 1: Possible roots

By Rational Root Theorem: possible roots = $\pm 1, \pm 2, \pm 3, \pm 6$.

Step 2: Test values

- $Q(1) = 1 - 6 + 11 - 6 = 0 \to (x - 1)$ is a factor.

Step 3: Divide by $(x - 1)$

Synthetic division:

$$
\begin{array}{r|rrrr}
1 & 1 & -6 & 11 & -6 \\
\hline
 & & 1 & -5 & 6 \\
 & 1 & -5 & 6 & 0
\end{array}
$$

Quotient: $x^2 - 5x + 6$. Factor: $(x - 2)(x - 3)$.

Final Answer: $Q(x) = (x - 1)(x - 2)(x - 3)$

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Worksheet

Part A: Synthetic Division Practice

1. Divide $P(x) = 2x^3 + 5x^2 - 4x - 3$ by $(x+1)$. 2. Divide $Q(x) = x^3 - 7x^2 + 14x - 8$ by $(x-2)$. 3. Divide $R(x) = x^4 - x^3 - 7x^2 + x + 6$ by $(x-3)$.

Part B: Factoring with Rational Root Theorem

4. Factor $S(x) = 2x^3 - 3x^2 - 8x + 12$. 5. Factor $T(x) = x^3 + 2x^2 - 5x - 6$.
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Answer Key (Step by Step)

Part A

1. $P(x) = 2x^3 + 5x^2 - 4x - 3$, divide by $(x+1) \Rightarrow c = -1$.
Synthetic division:

$$\begin{array}{r|rrrr} -1 & 2 & 5 & -4 & -3 \\ \hline & & -2 & -3 & 7 \\ & 2 & 3 & -7 & 4 \end{array}$$

Remainder $= 4$, quotient $= 2x^2 + 3x - 7$.
2. $Q(x) = x^3 - 7x^2 + 14x - 8$, divide by $(x-2)$.

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 14 & -8 \\ \hline & & 2 & -10 & 8 \\ & 1 & -5 & 4 & 0 \end{array}$$

Quotient $= x^2 - 5x + 4 = (x-1)(x-4)$.
3. $R(x) = x^4 - x^3 - 7x^2 + x + 6$, divide by $(x-3)$.

$$\begin{array}{r|rrrrr} 3 & 1 & -1 & -7 & 1 & 6 \\ \hline & & 3 & 6 & -3 & -6 \\ & 1 & 2 & -1 & -2 & 0 \end{array}$$

Quotient $= x^3 + 2x^2 - x - 2$.
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Part B

4. $S(x) = 2x^3 - 3x^2 - 8x + 12$.

Possible roots: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{2}, \pm\frac{3}{2}$. Test $x = 2$: $2(8) - 3(4) - 16 + 12 = 0$. Divide using synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -8 & 12 \\ \hline & & 4 & 2 & -12 \\ & 2 & 1 & -6 & 0 \end{array}$$

Quotient $= 2x^2 + x - 6 = (2x - 3)(x + 2)$.

Final Answer: $(x - 2)(2x - 3)(x + 2)$

5. $T(x) = x^3 + 2x^2 - 5x - 6$.

Possible roots: $\pm1, \pm2, \pm3, \pm6$. Test $x = -1$: $-1 + 2 + 5 - 6 = 0$. Divide using synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ \hline & & -1 & -1 & 6 \\ & 1 & 1 & -6 & 0 \end{array}$$

Quotient $= x^2 + x - 6 = (x + 3)(x - 2)$.

Final Answer: $(x + 1)(x + 3)(x - 2)$