Lesson 3: Inverse Functions

ApexMath

Notes

1) Definition of an Inverse Function

- The inverse of a function f, denoted f^{-1}, reverses the input-output relationship of f.
- Each point (a, b) on the graph of f becomes (b, a) on the graph of f^{-1}.
- For a function $f(x)$:
$$f(x) = y \quad \Rightarrow \quad f^{-1}(y) = x$$
- Important: Not every function has an inverse. A function must be **one-to-one**.

2) Vertical Line Test and Horizontal Line Test

- Vertical line test: If any vertical line crosses the graph more than once, it is **not** a function.
- Horizontal line test: If any horizontal line crosses the graph more than once, the function does **not** have an inverse that is also a function.

3) Key Property of Inverse Functions

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

This means applying a function and then its inverse (or vice versa) returns the original input.

4) Steps to Find the Inverse of a Function

1. Replace $f(x)$ with y.
2. Swap x and y.
3. Solve the new equation for y.
4. Replace y with $f^{-1}(x)$.
5. Include domain restrictions if needed.

5) Domain and Range

- The domain of f becomes the range of f^{-1}, and the range of f becomes the domain of f^{-1}.
- Always check for restrictions to ensure the inverse is a function.

Walkthrough of Example Questions

Example 1: Linear Function

Find the inverse of $f(x) = 3x - 5$.

$$y = 3x - 5$$

Swap x and y (input-output swap):

$$x = 3y - 5$$

Solve for y:

$$y = \frac{x+5}{3}$$

So,

$$f^{-1}(x) = \frac{x+5}{3}$$

Verify the property:

$$f(f^{-1}(x)) = 3\left(\frac{x+5}{3}\right) - 5 = x$$

$$f^{-1}(f(x)) = \frac{(3x-5)+5}{3} = x$$

Example 2: Quadratic Function (Domain Restriction Required)

Find the inverse of $f(x) = x^2$, $x \geq 0$.

$$y = x^2$$

Swap x and y:

$$x = y^2$$

Solve for y:

$$y = \sqrt{x} \quad (\text{since } x \geq 0)$$

$$f^{-1}(x) = \sqrt{x}, \quad x \geq 0$$

Worksheet

A. Find the Inverse Functions

1. $f(x) = 2x + 7$
2. $f(x) = \frac{x-4}{5}$
3. $f(x) = x^2 + 3,\ x \geq 0$
4. $f(x) = \sqrt{x-2}$
5. $f(x) = \frac{1}{x+1},\ x \neq -1$

B. Domain and Range

1. Determine the domain and range of $f^{-1}(x)$ for each function above.

C. Application Problem

The temperature in Celsius T_C is converted to Fahrenheit T_F by $T_F = \frac{9}{5}T_C + 32$. Find the inverse function and explain what it represents. Then, determine the Celsius temperature when Fahrenheit is 77.

Answer Key

A. Inverse Functions

1. $f(x) = 2x + 7$

$$y = 2x + 7 \implies x = 2y + 7 \implies y = \frac{x-7}{2}$$

$$\boxed{f^{-1}(x) = \frac{x-7}{2}}$$

2. $f(x) = \frac{x-4}{5}$

$$y = \frac{x-4}{5} \implies x = \frac{y-4}{5} \implies y = 5x + 4$$

$$\boxed{f^{-1}(x) = 5x + 4}$$

3. $f(x) = x^2 + 3,\ x \geq 0$

$$y = x^2 + 3 \implies x = y^2 + 3 \implies y = \sqrt{x-3}$$

Domain: $x \geq 3$

$$\boxed{f^{-1}(x) = \sqrt{x-3},\ x \geq 3}$$

4. $f(x) = \sqrt{x-2}$

$$y = \sqrt{x-2} \implies x = \sqrt{y-2} \implies y = x^2 + 2$$

Domain: $x \geq 0$

$$\boxed{f^{-1}(x) = x^2 + 2,\ x \geq 0}$$

5. $f(x) = \frac{1}{x+1}, x \neq -1$

$$y = \frac{1}{x+1} \implies x = \frac{1}{y+1} \implies xy + x = 1 \implies y = \frac{1-x}{x}$$

Domain: $x \neq 0$

$$\boxed{f^{-1}(x) = \frac{1-x}{x},\ x \neq 0}$$

B. Domain and Range

1. $f^{-1}(x) = \frac{x-7}{2} \rightarrow$ Domain and range: all real numbers.
2. $f^{-1}(x) = 5x + 4 \rightarrow$ Domain and range: all real numbers.
3. $f^{-1}(x) = \sqrt{x-3} \rightarrow$ Domain: $x \geq 3$, Range: $y \geq 0$
4. $f^{-1}(x) = x^2 + 2 \rightarrow$ Domain: $x \geq 0$, Range: $y \geq 2$
5. $f^{-1}(x) = \frac{1-x}{x} \rightarrow$ Domain: $x \neq 0$, Range: $y \neq -1$

C. Application Problem

Original function: $T_F = \frac{9}{5}T_C + 32$ Swap and solve:

$$x = \frac{9}{5}y + 32 \implies y = \frac{5}{9}(x - 32)$$

Inverse function: converts Fahrenheit to Celsius.
Determine Celsius for $T_F = 77$:

$$T_C = \frac{5}{9}(77 - 32) = 25$$

$$\boxed{T_C = 25^\circ\text{C}}$$