Lesson 4: Radical and Rational Functions

ApexMath

Notes

1) Radical Functions

- Radical functions involve roots, e.g., $f(x) = \sqrt{x}$ or $f(x) = \sqrt[3]{x-2}$.
- Radicand: The expression inside the radical, e.g., in $\sqrt{x-2}$ the radicand is $x - 2$.
- Index: The number of the root. $\sqrt{x}$ has index 2 (square root), $\sqrt[3]{x}$ has index 3 (cube root).
- Domain: For even-index radicals (square roots, 4th roots, etc.), the radicand ≥ 0. Odd-index radicals can take all real numbers.
- Range: Even roots output $y \geq 0$; odd roots output all real numbers.

2) Rational Functions

- Rational functions are fractions of polynomials, e.g., $f(x) = \frac{x+2}{x-3}$.
- Domain: All x except where denominator $= 0$.
- Vertical asymptotes: Values of x where the denominator $= 0$ and numerator $\neq 0$.
- Horizontal asymptotes: Compare the degree of the numerator (n) and the degree of the denominator (m):
  - If $n < m$, horizontal asymptote: $y = 0$.
  - If $n = m$, horizontal asymptote: ratio of leading coefficients of numerator and denominator.
  - If $n > m$, no horizontal asymptote (may have an oblique asymptote). An oblique asymptote is a slanted line the graph approaches as $x \to \pm\infty$.

3) Operations with Radical and Rational Functions

- Add, subtract, multiply, divide as usual. Remember domain restrictions.
- Simplify radicals and reduce rational expressions.

4) Inverse Functions and Domain Restrictions

- Some radical and rational functions require domain restrictions to have inverses.
- Swap $(x, y) \to (y, x)$ and solve for y to find the inverse.
- Always check that the inverse's domain makes sense.
- Remember the rule $f(f^{-1}(x)) = x$.

Walkthrough of Example Questions

Example 1: Radical Function

Find the domain, range, and inverse of $f(x) = \sqrt{x-5}$.

Step 1: Domain

Radicand $x - 5 \geq 0 \implies x \geq 5$

Step 2: Range

Square root outputs $y \geq 0$

Step 3: Inverse

$$y = \sqrt{x-5} \implies x = \sqrt{y-5} \quad \text{(swap } x \text{ and } y\text{)}$$
$$x^2 = y - 5 \implies y = x^2 + 5$$

Domain of f^{-1}: $x \geq 0$, Range: $y \geq 5$

Example 2: Rational Function

Find the domain and asymptotes of $f(x) = \frac{2x+1}{x-3}$.

Step 1: Domain

Denominator $\neq 0 \implies x \neq 3$

Step 2: Vertical asymptote

Set denominator $= 0$: $x - 3 = 0 \implies x = 3$

Step 3: Horizontal asymptote

Degrees numerator $=$ denominator $= 1 \to y = \frac{2}{1} = 2$

Step 4: Inverse

$$y = \frac{2x+1}{x-3} \implies y(x-3) = 2x + 1 \implies yx - 3y = 2x + 1$$
$$yx - 2x = 3y + 1 \implies x(y-2) = 3y + 1 \implies x = \frac{3y+1}{y-2}$$

$$f^{-1}(x) = \frac{3x+1}{x-2}, \quad x \neq 2$$

Worksheet

A. Radical Functions

1. $f(x) = \sqrt{x+4}$
2. $f(x) = \sqrt[3]{2x-1}$
3. $f(x) = \sqrt{3-x}$
4. $f(x) = \sqrt[4]{x-2}$

B. Rational Functions

1. $f(x) = \frac{x-2}{x+5}$
2. $f(x) = \frac{3x+1}{2x-4}$
3. $f(x) = \frac{x^2-1}{x-1}$
4. $f(x) = \frac{x^2+2x+1}{x-3}$

C. Real-Life Application

A car travels along a straight road and its velocity is modeled by $v(t) = \frac{10t}{t+2}$ m/s.

1) Find the domain. 2) Find the inverse function and interpret it. 3) Determine the time when the car reaches 5 m/s.

Answer Key

A. Radical Functions

1. $f(x) = \sqrt{x+4}$ Domain: $x \geq -4$, Range: $y \geq 0$ Inverse: $f^{-1}(x) = x^2 - 4, x \geq 0$

   $$\boxed{f^{-1}(x) = x^2 - 4,\ x \geq 0}$$

2. $f(x) = \sqrt[3]{2x-1}$ Domain and range: all real numbers Inverse: $f^{-1}(x) = \frac{x^3+1}{2}$

   $$\boxed{f^{-1}(x) = \frac{x^3+1}{2}}$$

3. $f(x) = \sqrt{3-x}$ Domain: $x \leq 3$, Range: $y \geq 0$ Inverse: $f^{-1}(x) = 3 - x^2, x \geq 0$

   $$\boxed{f^{-1}(x) = 3 - x^2, x \geq 0}$$

4. $f(x) = \sqrt[4]{x-2}$ Domain: $x \geq 2$, Range: $y \geq 0$ Inverse: $f^{-1}(x) = x^4 + 2, x \geq 0$

$$\boxed{f^{-1}(x) = x^4 + 2, x \geq 0}$$

B. Rational Functions

1. $f(x) = \frac{x-2}{x+5}$ Domain: $x \neq -5$, VA: $x = -5$, HA: $y = 1$, Inverse: $f^{-1}(x) = \frac{5x+2}{1-x}, x \neq 1$
2. $f(x) = \frac{3x+1}{2x-4}$ Domain: $x \neq 2$, VA: $x = 2$, HA: $y = 3/2$, Inverse: $f^{-1}(x) = \frac{4x+1}{3-2x}, x \neq 3/2$
3. $f(x) = \frac{x^2-1}{x-1} = x + 1, x \neq 1$ Domain: $x \neq 1$, Inverse: $f^{-1}(x) = x - 1$
4. $f(x) = \frac{x^2+2x+1}{x-3}$ Domain: $x \neq 3$, VA: $x = 3$, HA: Since degree numerator $(n = 2)$ ¿ denominator $(m = 1) \rightarrow$ no HA, oblique asymptote exists Inverse: Solve $y(x-3) = x^2 + 2x + 1 \implies x^2 + 2x + 1 - xy + 3y = 0$ (quadratic in x)

C. Real-Life Application

$v(t) = \frac{10t}{t+2}$

1) Domain: $t \geq 0$ 2) Inverse: $v = \frac{10t}{t+2} \implies t = \frac{-2v}{v-10}$ 3) Solve for $v = 5$: $t = \frac{-2(5)}{5-10} = 2$ s

$$\boxed{t = 2 \text{ s}}$$