Functions & Relations Mastery Pack — Review Lesson

ApexMath

Notes

1) Introduction to Functions

A **function** matches each input x to exactly one output y. **Vertical line test:** If any vertical line crosses the graph more than once, it is **not** a function. **Domain** is the set of all x-values we are allowed to use. **Range** is the set of all possible outputs $f(x)$.

2) Function Operations

If f and g are functions, then

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x),$$
$$(f \cdot g)(x) = f(x)g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ with } g(x) \neq 0.$$

Always check domain restrictions (for example, no division by 0, no square roots of negative numbers, etc.).

3) Inverse Functions

The inverse "undoes" the original function. If (a,b) lies on $y = f(x)$, then (b,a) lies on $y = f^{-1}(x)$. The key rule is:

$$f\big(f^{-1}(x)\big) = x \quad \text{and} \quad f^{-1}\big(f(x)\big) = x \text{ (when both sides are defined).}$$

4) Radical Functions

A **radicand** is the expression inside the radical sign. The **index** is the small number that tells which root ($\sqrt[n]{\cdot}$). For square roots ($\sqrt{\cdot}$), the radicand must be ≥ 0 for real outputs. For odd roots (like $\sqrt[3]{\cdot}$), any real number is allowed.

5) Factoring Higher-Degree Polynomials

Use zeros and **multiplicity**:

- If a zero $x = r$ has even multiplicity, the graph touches the x-axis and turns around.
- If it has odd multiplicity, the graph crosses the x-axis.

End behavior comes from the leading term and degree (even degree: both ends same direction; odd degree: ends opposite). **Factor Theorem:** $(x - c)$ is a factor of $P(x)$ if and only if $P(c) = 0$. **Remainder Theorem:** Dividing $P(x)$ by $(x - c)$ leaves remainder $P(c)$. Synthetic division can help test and factor efficiently.

6) Graph Interpretation Using Multiplicity

From the graph: locate zeros, note whether the graph crosses (odd multiplicity) or just touches (even multiplicity), and infer the leading sign from end behavior.

7) Applications of Functions

- **Trigonometric:** cycles (e.g., tides, Ferris wheels) with amplitude, period, phase shift, and vertical shift.
- **Exponential:** growth/decay (e.g., populations, medicine levels), often solved with logarithms.
- **Sequences/Series:** repeated patterns (arithmetic: constant difference; geometric: constant ratio).

Walkthroughs (Step-by-Step)

Walkthrough A: Trigonometric Application (Height on a Wheel)

Problem. A wheel has radius 12 m. The lowest seat is 3 m above ground. One rotation takes 30 s. At $t = 0$, the rider is at the lowest point. **(i)** Build $h(t)$. **(ii)** When is $h(t) = 21$ m for the first time?

Step 1 (Amplitude and midline). Amplitude $A = 12$. Midline $D = 12 + 3 = 15$. **Step 2 (Period and B).** $T = 30 \Rightarrow B = \dfrac{2\pi}{T} = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$. **Step 3 (Phase/starting position).** At $t = 0$ at the lowest point. Using cosine, the lowest occurs if we use $-A\cos(Bt)$ (since $\cos 0 = 1$):

$$\boxed{h(t) = -12\cos\left(\tfrac{\pi}{15}t\right) + 15}$$

Step 4 (Solve $h(t) = 21$).

$$-12\cos\left(\tfrac{\pi}{15}t\right) + 15 = 21 \;\Rightarrow\; -12\cos(\cdot) = 6 \;\Rightarrow\; \cos(\cdot) = -0.5.$$

The first positive solution for $\cos\theta = -\frac{1}{2}$ is $\theta = \dfrac{2\pi}{3}$. Hence

$$\tfrac{\pi}{15}t = \frac{2\pi}{3} \;\Rightarrow\; t = \frac{2\pi}{3}\cdot\frac{15}{\pi} = 10 \text{ s.}$$

$$\boxed{t = 10 \text{ s (first time)}}$$

(Another time in the same rotation is at $\theta = \frac{4\pi}{3}$, giving $t = 20$ s.)

Walkthrough B: Exponential Decay (Medicine in Bloodstream)

Problem. The concentration (mg) after t hours is $C(t) = 200e^{-0.25t}$. **(i)** When does $C(t)$ reach 50 mg? **(ii)** What is $C(3)$?

Step (i).

$$200e^{-0.25t} = 50 \;\Rightarrow\; e^{-0.25t} = 0.25 \;\Rightarrow\; -0.25t = \ln(0.25).$$

Since $\ln(0.25) = -\ln 4$,

$$t = \frac{\ln 4}{0.25} \approx \frac{1.3863}{0.25} \approx \boxed{5.55 \text{ hours}}.$$

Step (ii).

$$C(3) = 200e^{-0.75} \approx 200(0.47237) \approx \boxed{94.5 \text{ mg (approx)}}.$$

Walkthrough C: Arithmetic Sequence (Weekly Students)

Problem. You tutor 6 students in week 1 and gain 3 more students each week. **(i)** How many students in week 15? **(ii)** Total students taught over the first 15 weeks?

Step (i). Arithmetic nth term: $a_n = a_1 + (n-1)d$.

$$a_{15} = 6 + (15-1)\cdot 3 = 6 + 42 = \boxed{48}.$$

Step (ii). Sum: $S_n = \dfrac{n}{2}\big(2a_1 + (n-1)d\big)$.

$$S_{15} = \frac{15}{2}\big(2\cdot 6 + 14\cdot 3\big) = \frac{15}{2}(12+42) = \frac{15}{2}\cdot 54 = \boxed{405}.$$

Walkthrough D: Geometric Sequence (Bouncing Ball Distance)

Problem. A ball is dropped from 9 m and rebounds to 60% of the previous height. Find the total distance traveled by the time it hits the ground for the 5th time.

Heights (down then up): first drop: 9; then up & down bounces form a geometric series with first term $2\cdot 9\cdot 0.6$ and ratio 0.6. Total distance after 5 bounces:

$$D = 9 + 2\cdot 9\cdot 0.6\cdot\frac{1-0.6^5}{1-0.6}.$$

Compute $0.6^5 = 0.07776$:

$$D = 9 + 10.8\cdot\frac{1-0.07776}{0.4} = 9 + 10.8\cdot\frac{0.92224}{0.4} = 9 + 10.8\cdot 2.3056 \approx 9 + 24.90048.$$

$$\boxed{D \approx 33.90 \text{ m}}.$$

Worksheet

1. **Trigonometric Model (Tides).** In a harbor, the water depth ranges from a low of 1.5 m to a high of 5.5 m and follows a sinusoidal pattern with a period of 12.4 hours. At $t = 0$, the depth is at its <u>minimum</u> and rising.

   (a) Build a depth model $d(t)$.

   (b) Find the first time t when $d(t) = 4.3$ m.

2. **Exponential Growth (Savings).** Your balance grows with continuous compounding: $A(t) = 800e^{0.045t}$, where t is in years.

   (a) How long until the balance reaches \$1000?

   (b) What is $A(6)$?

3. **Exponential Decay (Cooling).** A hot drink cools following $T(t) = 22 + 58e^{-0.4t}$ (temperature in °C, room is 22°C).

   (a) When will the drink reach 30°C?

   (b) What is the temperature after 10 minutes?

4. **Arithmetic Sequence (Tiles).** A staircase has rows of tiles. The first row has 12 tiles, and each new row has 5 more than the last.

   (a) How many tiles are in row 20?

   (b) How many tiles are used in the first 20 rows?

5. **Geometric Sequence (Bounces).** A ball is dropped from 7 m and rebounds to 65% of the previous height.

   (a) Write a formula for the rebound heights.

   (b) Find the total distance traveled by the time it hits the ground for the 6th time.

6. **Inverse Functions.** Let $f(x) = \dfrac{3x - 4}{5}$.

   (a) Find $f^{-1}(x)$.

   (b) Verify that $f\big(f^{-1}(x)\big) = x$.

   (c) If $(2, f(2))$ lies on $y = f(x)$, what point lies on $y = f^{-1}(x)$?

7. **Radical & Rational Domains.**

   (a) $g(x) = \sqrt{2x - 6}$: state the domain.

   (b) $h(x) = \dfrac{\sqrt[3]{x - 5}}{x - 4}$: state the domain.

8. **Function Operations & Domain.** Let $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 - 9$.

(a) Find $(f+g)(x)$ and its domain.

(b) Find $\left(\frac{f}{g}\right)(x)$ and its domain.

9. **Graph Interpretation & Multiplicity.** A polynomial has zeros at $x=-2$ (mult 2), $x=1$ (mult 1), and $x=4$ (mult 1). The leading coefficient is negative.

   (a) Write a possible formula for $P(x)$ (up to a positive constant k).

   (b) State the end behavior as $x \to \pm\infty$.

Answer Key (Detailed, Step-by-Step)

1. Trigonometric Model (Tides)

Max = 5.5, Min = 1.5. Amplitude $A = \frac{5.5-1.5}{2} = 2.0$. Midline $D = \frac{5.5+1.5}{2} = 3.5$. Period $T = 12.4 \Rightarrow B = \frac{2\pi}{12.4} = \frac{\pi}{6.2}$. At $t=0$ the depth is minimum and rising, so a cosine model with negative amplitude works:

$$\boxed{d(t) = -2\cos\left(\tfrac{\pi}{6.2}t\right) + 3.5}.$$

Solve $-2\cos(\frac{\pi}{6.2}t) + 3.5 = 4.3$:

$$-2\cos(\cdot) = 0.8 \Rightarrow \cos(\cdot) = -0.4.$$

First positive solution is $\theta = \cos^{-1}(-0.4) \approx 1.9823$ (radians). Then

$$t = \frac{1.9823}{\pi/6.2} \approx \frac{1.9823 \cdot 6.2}{3.1416} \approx \boxed{3.91 \text{ hours}}.$$

2. Exponential Growth (Savings)

$A(t) = 800e^{0.045t}$. (a) $800e^{0.045t} = 1000 \Rightarrow e^{0.045t} = 1.25 \Rightarrow 0.045t = \ln(1.25)$.

$$t = \frac{\ln(1.25)}{0.045} \approx \frac{0.22314}{0.045} \approx \boxed{4.96 \text{ years}}.$$

(b) $A(6) = 800e^{0.27} \approx 800(1.310) = \boxed{1{,}048 \text{ (approx)}}$.

3. Exponential Decay (Cooling)

$T(t) = 22 + 58e^{-0.4t}$. (a) $22 + 58e^{-0.4t} = 30 \Rightarrow 58e^{-0.4t} = 8 \Rightarrow e^{-0.4t} = \frac{8}{58} = \frac{4}{29}$.

$$-0.4t = \ln\left(\frac{4}{29}\right) \Rightarrow t = \frac{-\ln(4/29)}{0.4} = \frac{\ln(29/4)}{0.4} \approx \frac{1.9741}{0.4} \approx \boxed{4.94 \text{ min}}.$$

(b) $T(10) = 22 + 58e^{-4} \approx 22 + 58(0.01832) \approx 22 + 1.062 \approx \boxed{23.06°\text{C}}$.

4. Arithmetic Sequence (Tiles)

$a_1 = 12$, $d = 5$. (a) $a_{20} = 12 + (20-1)\cdot 5 = 12 + 95 = \boxed{107}$. (b) $S_{20} = \frac{20}{2}(12+107) = 10 \cdot 119 = \boxed{1190}$.

5. Geometric Sequence (Bounces)

Heights: 7, $7 \cdot 0.65$, $7 \cdot 0.65^2, \ldots$ Total distance to 6th ground hit:

$$D = 7 + 2 \cdot 7 \cdot 0.65 \cdot \frac{1 - 0.65^6}{1 - 0.65}.$$

Compute $0.65^6 \approx 0.07542$. Then

$$D = 7 + 9.1 \cdot \frac{1 - 0.07542}{0.35} = 7 + 9.1 \cdot \frac{0.92458}{0.35} = 7 + 9.1 \cdot 2.64166 \approx 7 + 24.64 \approx \boxed{31.64 \text{ m}}.$$

6. Inverse Functions

$f(x) = \frac{3x-4}{5}$. (a) Set $y = \frac{3x-4}{5} \Rightarrow 5y = 3x - 4 \Rightarrow 3x = 5y + 4 \Rightarrow x = \frac{5y+4}{3}$. Swap (x, y) to get $f^{-1}(x) = \frac{5x+4}{3}$:

$$\boxed{f^{-1}(x) = \frac{5x+4}{3}}.$$

(b) Verify:

$$f\big(f^{-1}(x)\big) = \frac{3 \cdot \frac{5x+4}{3} - 4}{5} = \frac{5x + 4 - 4}{5} = \frac{5x}{5} = x.$$

(c) $f(2) = \frac{6-4}{5} = \frac{2}{5} = 0.4$. So $(2, 0.4)$ is on $y = f(x)$, hence $(0.4, 2)$ is on $y = f^{-1}(x)$:

$$\boxed{\text{Point on } f^{-1}:\ (0.4,\ 2)}.$$

7. Radical & Rational Domains

(a) $g(x) = \sqrt{2x-6}$ requires $2x - 6 \geq 0 \Rightarrow x \geq 3$:

$$\boxed{\text{Domain of } g : [3, \infty)}.$$

(b) $h(x) = \frac{\sqrt[3]{x-5}}{x-4}$. Cube roots allow any real $x - 5$, but denominator $\neq 0$. Thus

$$\boxed{\text{Domain of } h :\ \mathbb{R} \setminus \{4\}}.$$

8. Function Operations & Domain

$f(x) = \sqrt{x-1}$ (needs $x \geq 1$), $g(x) = x^2 - 9$. (a) $(f+g)(x) = x^2 - 9 + \sqrt{x-1}$, with $x \geq 1$:

$$\boxed{(f+g)(x) = x^2 - 9 + \sqrt{x-1},\ \text{domain } [1, \infty)}.$$

(b) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-1}}{x^2-9}$, with $x \geq 1$ and $x^2 - 9 \neq 0$. So $x \neq 3$ (and $x = -3$ is already < 1). Domain: $[1, \infty)$ but $x \neq 3$:

$$\boxed{\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-1}}{x^2-9},\ \text{domain } [1, \infty) \setminus \{3\}}.$$

9. Graph Interpretation & Multiplicity

Zeros: -2 (mult 2, touch), 1 (mult 1, cross), 4 (mult 1, cross). Leading coefficient < 0 and total degree 4 (even) $\Rightarrow$ both ends down. A possible form (with $k > 0$):

$$\boxed{P(x) = -k(x+2)^2(x-1)(x-4)}.$$

End behavior: as $x \to \pm\infty$, $P(x) \to -\infty$:

$$\boxed{\text{Down on both ends.}}$$