Reciprocal Functions
ApexMath

Notes

What Is a Reciprocal Function?

The **parent reciprocal function** is:

$$y = \frac{1}{x}$$

Unlike lines, parabolas, or absolute value functions, this function has a value that is **undefined at** $x = 0$ (you cannot divide by zero). As x gets closer to 0, the output grows without bound. As x gets very large, the output approaches 0 but never reaches it.

This produces a graph with **two branches** — one in the first quadrant and one in the third quadrant — that curve toward two invisible boundary lines called **asymptotes**.

Asymptotes

An **asymptote** is a line that the graph approaches but *never touches*.

- The **vertical asymptote (VA)** is the value of x that makes the denominator equal to zero. For $y = \frac{1}{x}$, the VA is $x = 0$ (the y-axis).
- The **horizontal asymptote (HA)** is the value that y approaches as $x \to \pm\infty$. For $y = \frac{1}{x}$, the HA is $y = 0$ (the x-axis).

Both asymptotes are drawn as **dashed lines** on the graph.

The Transformed Form

Every transformed reciprocal function can be written as:

$$\boxed{y = \frac{1}{x - h} + k}$$

- h shifts the graph **horizontally**: the vertical asymptote moves to $x = h$.
- k shifts the graph **vertically**: the horizontal asymptote moves to $y = k$.
- The two branches always sit in the two quadrants created by the intersection of the asymptotes.

Reading the Asymptotes Directly

For $y = \frac{1}{x - h} + k$:

Vertical asymptote:	$x = h$	(value that makes the denominator zero)
Horizontal asymptote:	$y = k$	(the constant added outside the fraction)

Domain and Range

- **Domain:** all real numbers *except* $x = h$. Written: $(-\infty, h) \cup (h, \infty)$.
- **Range:** all real numbers *except* $y = k$. Written: $(-\infty, k) \cup (k, \infty)$.

Pro Tip

- Always draw the asymptotes as dashed lines *first*, before sketching the curve. The two branches live in two of the four regions created by the crossing asymptotes. Never let the curve touch or cross an asymptote.
- To find the VA, ask: "what value of x makes the denominator equal zero?" Set $x-h = 0$ and solve. That gives $x = h$.
- To find the HA, look at the constant term added *outside* the fraction. That constant is k, and the HA is $y = k$.
- To sketch the branches quickly: plug in $x = h + 1$ and $x = h - 1$ to get one point on each branch, then sketch the hyperbola shape through each point approaching the asymptotes.
- Domain and range always exclude exactly one value each — the VA value from the domain and the HA value from the range. If you can state the asymptotes, you can immediately state domain and range.

Examples

Example 1 (The Parent Function): Sketch $y = \dfrac{1}{x}$ and state the asymptotes, domain, and range.

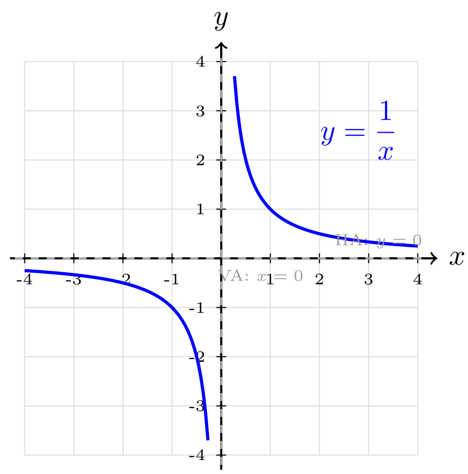
Step 1: Find the asymptotes.

The denominator is x. It equals zero when $x = 0$, so the **VA is** $x = 0$. There is no constant added outside, so the **HA is** $y = 0$.

Step 2: Build a table of values.

x	$y = 1/x$
-4	-0.25
-2	-0.5
-1	-1
1	1
2	0.5
4	0.25

Step 3: Draw asymptotes, then sketch both branches.



Features: VA: $x = 0$. HA: $y = 0$. Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$.

Example 2 (Horizontal Shift): Sketch $y = \dfrac{1}{x-3}$ and state all features.

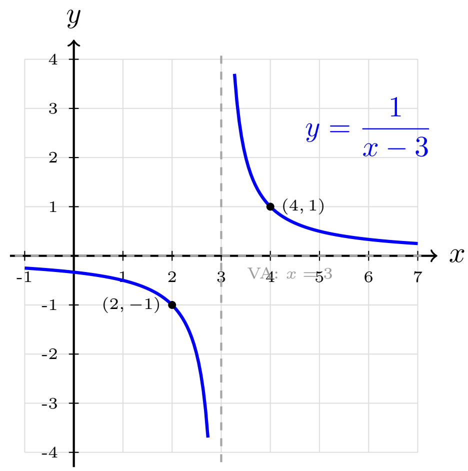
Step 1: Identify h and k.

$h = 3$, $k = 0$. VA: $x = 3$. HA: $y = 0$.

Step 2: Find key points. Plug in $x = h + 1 = 4$ and $x = h - 1 = 2$:

$$x = 4: \quad y = \frac{1}{4-3} = 1 \qquad x = 2: \quad y = \frac{1}{2-3} = -1$$

Step 3: Sketch.



Features: VA: $x = 3$. HA: $y = 0$. Domain: $(-\infty, 3) \cup (3, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$.

Example 3 (Vertical Shift): Sketch $y = \dfrac{1}{x} + 2$ and state all features.

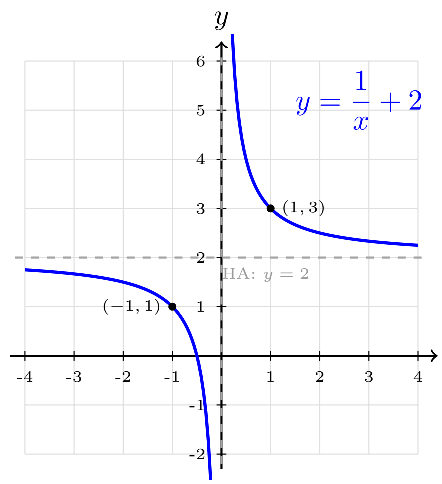
Step 1: Identify h and k.

$h = 0$, $k = 2$. VA: $x = 0$. HA: $y = 2$.

Step 2: Key points.

$$x = 1: \ y = \frac{1}{1} + 2 = 3 \qquad x = -1: \ y = \frac{1}{-1} + 2 = 1$$

Step 3: Sketch. The entire graph is the parent $y = \frac{1}{x}$ shifted **2 units up**. The HA rises from $y = 0$ to $y = 2$.



Features: VA: $x = 0$. HA: $y = 2$. Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, 2) \cup (2, \infty)$.

Example 4 (Combined Shift): Sketch $y = \dfrac{1}{x+2} - 1$ and state all features.

Step 1: Identify h and k.

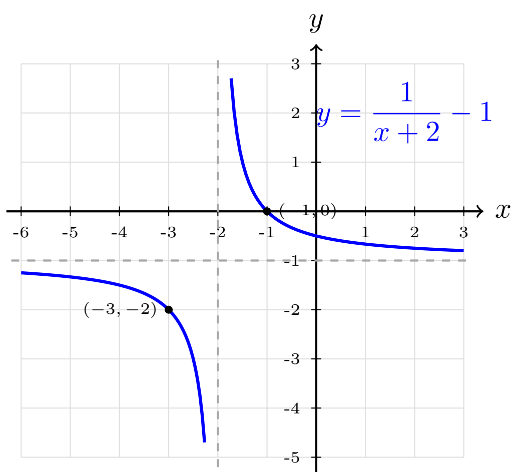
Rewrite as $y = \dfrac{1}{x-(-2)} + (-1)$, so $h = -2$, $k = -1$.

VA: $x = -2$. HA: $y = -1$.

Step 2: Key points.

$$x = -1: \; y = \frac{1}{-1+2} - 1 = 1 - 1 = 0 \qquad x = -3: \; y = \frac{1}{-3+2} - 1 = -1 - 1 = -2$$

Step 3: Sketch.



Features: VA: $x = -2$. HA: $y = -1$. Domain: $(-\infty, -2) \cup (-2, \infty)$. Range: $(-\infty, -1) \cup (-1, \infty)$.

Example 5 (Write the Equation): A reciprocal function has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 3$. Write its equation.

Step 1: VA at $x = 1$ means $h = 1$. HA at $y = 3$ means $k = 3$.

Step 2: Substitute into $y = \dfrac{1}{x-h} + k$.

$$y = \frac{1}{x-1} + 3$$

Step 3: Verify with a point.

$x = 2$: $y = \dfrac{1}{2-1} + 3 = 1 + 3 = 4$. Plotting $(2, 4)$ should lie on the right branch above the HA $y = 3$. ✓

$$\boxed{y = \frac{1}{x-1} + 3}$$

Common Mistakes

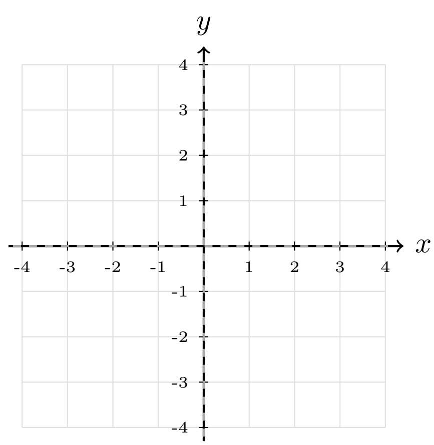
- **Touching or crossing the asymptote:** The curve never reaches or crosses an asymptote. If your sketch touches the dashed line, redraw it curving away toward the asymptote but never arriving.
- **Reversing the sign of h:** In $y = \dfrac{1}{x-3}$, the VA is $x = 3$, not $x = -3$. Set the denominator equal to zero: $x - 3 = 0$ gives $x = 3$.
- **Confusing h and k in domain and range:** The domain excludes h (the VA value) and the range excludes k (the HA value). Do not swap them.
- **Drawing only one branch:** The reciprocal function always has two separate branches. If you draw only one, the graph is incomplete.
- **Placing both branches in adjacent quadrants:** The two branches always sit in *opposite* quadrants relative to the asymptote intersection. If the right branch goes up-right, the left branch goes down-left — not up-left.

Worksheet

Part 1 — For each equation, state the VA, HA, domain, and range. Then sketch the graph on the provided axes.

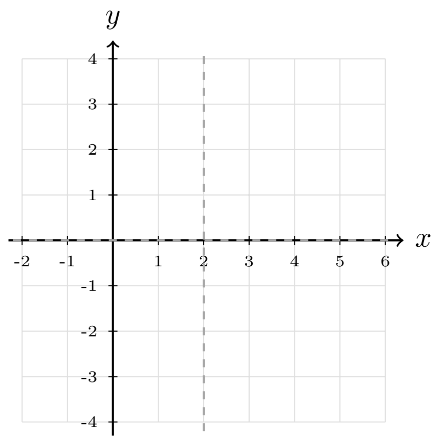
1. $y = \dfrac{1}{x}$

   VA: __________ HA: __________ Domain: __________________ Range: __________________



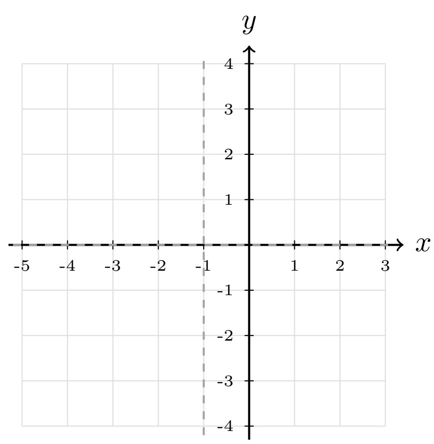
2. $y = \dfrac{1}{x-2}$

   VA: __________ HA: __________ Domain: __________________ Range: __________________



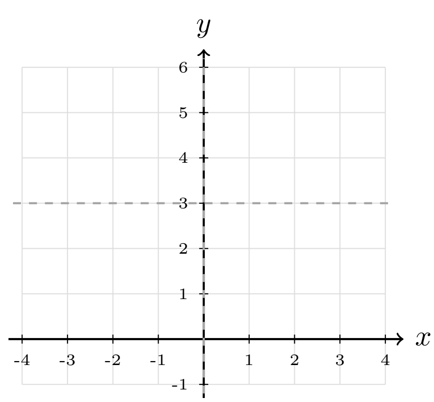
3. $y = \dfrac{1}{x+1}$

VA: ___________ HA: ___________ Domain: ___________________ Range: ___________________



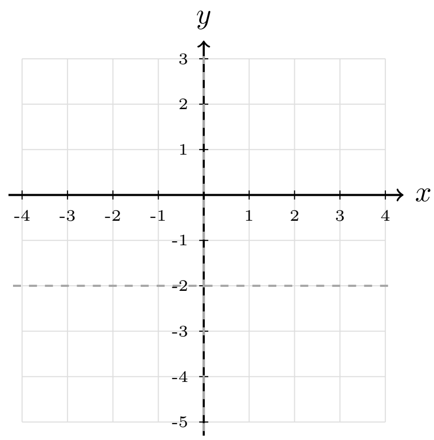
4. $y = \dfrac{1}{x} + 3$

VA: ___________ HA: ___________ Domain: ___________________ Range: ___________________



5. $y = \dfrac{1}{x} - 2$

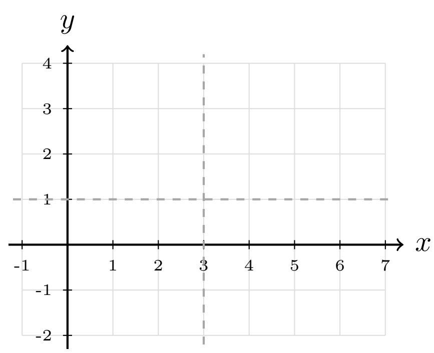
VA: ___________ HA: ___________ Domain: ___________________ Range: ___________________



Part 2 — State the VA, HA, domain, and range. Sketch the graph.

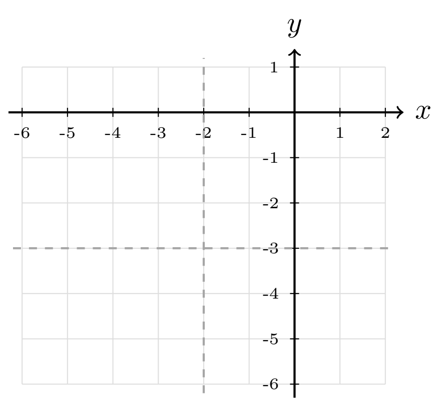
6. $y = \dfrac{1}{x-3} + 1$

   VA: ________ HA: ________ Domain: ______________ Range: ______________



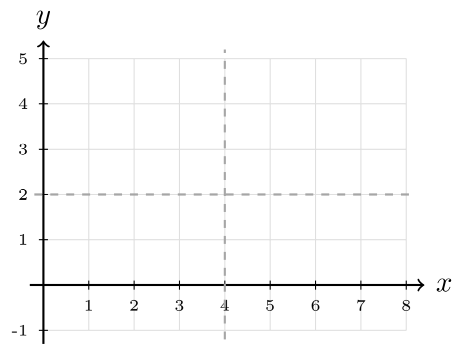
7. $y = \dfrac{1}{x+2} - 3$

   VA: ________ HA: ________ Domain: ______________ Range: ______________



8. $y = \dfrac{1}{x-4} + 2$

   VA: ________ HA: ________ Domain: ______________ Range: ______________



Part 3 — Challenge: Write the equation from the given information.

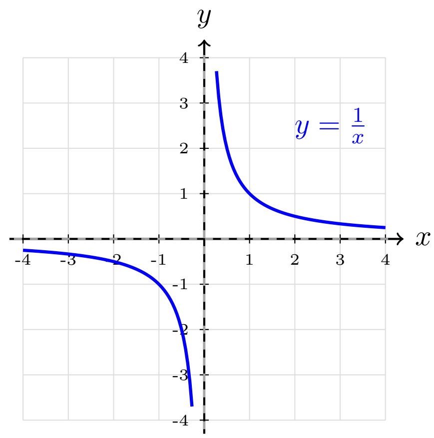
9. VA at $x = 1$, HA at $y = -1$.

10. VA at $x = -3$, HA at $y = 4$.

Answer Key

1. $y = \dfrac{1}{x}$

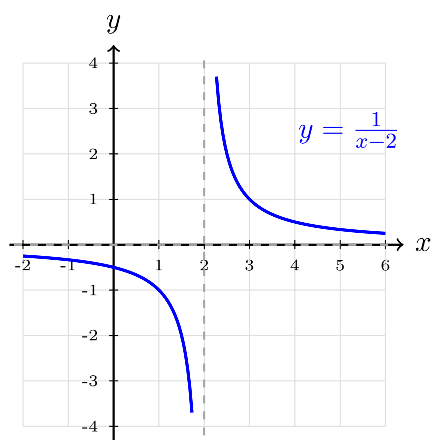
VA: $x = 0$. HA: $y = 0$. Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$.



2. $y = \dfrac{1}{x-2}$

VA: $x = 2$. HA: $y = 0$. Domain: $(-\infty, 2) \cup (2, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$.

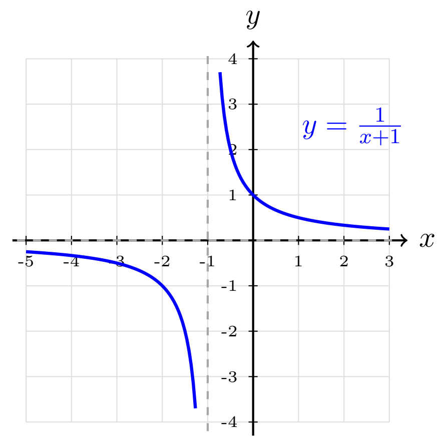
Key points: $x = 3 \Rightarrow y = 1$; $x = 1 \Rightarrow y = -1$.



3. $y = \dfrac{1}{x+1}$

VA: $x = -1$. HA: $y = 0$. Domain: $(-\infty, -1) \cup (-1, \infty)$. Range: $(-\infty, 0) \cup (0, \infty)$.

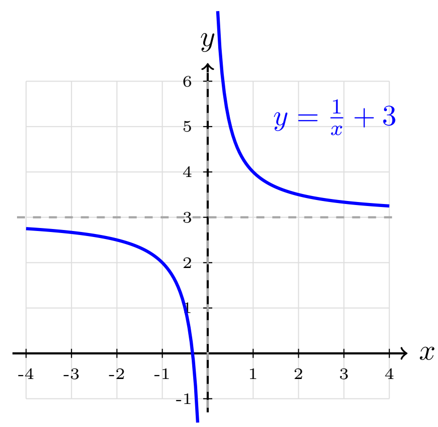
Key points: $x = 0 \Rightarrow y = 1$; $x = -2 \Rightarrow y = -1$.



4. $y = \dfrac{1}{x} + 3$

VA: $x = 0$. HA: $y = 3$. Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, 3) \cup (3, \infty)$.

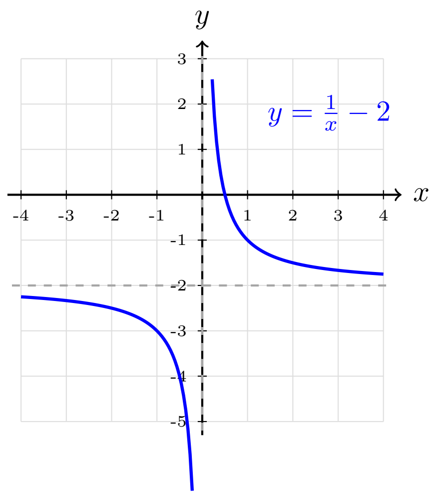
Key points: $x = 1 \Rightarrow y = 4$; $x = -1 \Rightarrow y = 2$.



5. $y = \dfrac{1}{x} - 2$

VA: $x = 0$. HA: $y = -2$. Domain: $(-\infty, 0) \cup (0, \infty)$. Range: $(-\infty, -2) \cup (-2, \infty)$.

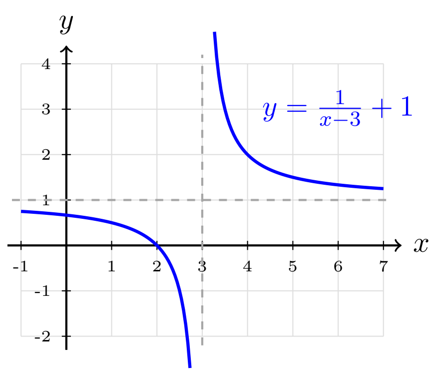
Key points: $x = 1 \Rightarrow y = -1$; $x = -1 \Rightarrow y = -3$.



6. $y = \dfrac{1}{x-3} + 1$

VA: $x = 3$. HA: $y = 1$. Domain: $(-\infty, 3) \cup (3, \infty)$. Range: $(-\infty, 1) \cup (1, \infty)$.

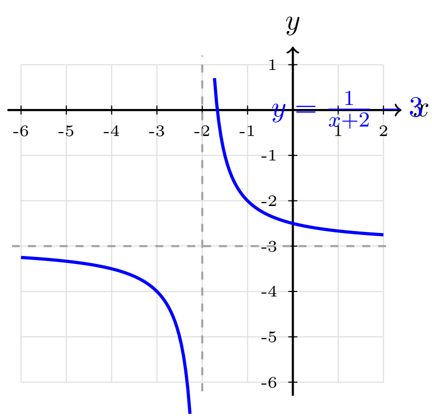
Key points: $x = 4 \Rightarrow y = 2$; $x = 2 \Rightarrow y = 0$.



7. $y = \dfrac{1}{x+2} - 3$

VA: $x = -2$. HA: $y = -3$. Domain: $(-\infty, -2) \cup (-2, \infty)$. Range: $(-\infty, -3) \cup (-3, \infty)$.

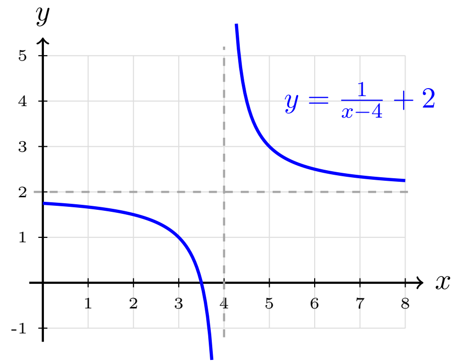
Key points: $x = -1 \Rightarrow y = -2$; $x = -3 \Rightarrow y = -4$.



8. $y = \dfrac{1}{x-4} + 2$

VA: $x = 4$. HA: $y = 2$. Domain: $(-\infty, 4) \cup (4, \infty)$. Range: $(-\infty, 2) \cup (2, \infty)$.

Key points: $x = 5 \Rightarrow y = 3$; $x = 3 \Rightarrow y = 1$.



9. VA at $x = 1$, HA at $y = -1$.

$h = 1$, $k = -1$. Substitute into $y = \dfrac{1}{x-h} + k$:

$$y = \frac{1}{x-1} + (-1) = \frac{1}{x-1} - 1$$

Verify: $x = 2 \Rightarrow y = 1 - 1 = 0$ ✓. $x = 0 \Rightarrow y = -1 - 1 = -2$ ✓.

$$\boxed{y = \frac{1}{x-1} - 1}$$

10. VA at $x = -3$, HA at $y = 4$.

$h = -3$, $k = 4$. Substitute into $y = \dfrac{1}{x-h} + k$:

$$y = \frac{1}{x-(-3)} + 4 = \frac{1}{x+3} + 4$$

Verify: $x = -2 \Rightarrow y = 1 + 4 = 5$ ✓. $x = -4 \Rightarrow y = -1 + 4 = 3$ ✓.

$$\boxed{y = \frac{1}{x+3} + 4}$$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson · ApexMath
