Solving Absolute Value Equations
ApexMath

Notes

The Core Idea: Two Distances

In Lesson 1 we learned that $|x|$ measures the distance of x from zero. When we write $|x| = 5$, we are asking: *which values of x are exactly 5 units from zero?* There are two such values: $x = 5$ and $x = -5$.

This is the key insight behind every absolute value equation: **one equation splits into two**.

The Split Rule

For any expression A and any non-negative number c:

$$|A| = c \qquad \Longrightarrow \qquad A = c \quad \textbf{or} \quad A = -c$$

Solve each case separately and collect all solutions.

The Three Cases

Before splitting, always check the right-hand side:

- **$c > 0$:** Two solutions. Split into $A = c$ and $A = -c$.
- **$c = 0$:** One solution. $|A| = 0$ means $A = 0$ — there is only one number whose distance from zero is zero.
- **$c < 0$:** No solution. Distance is never negative, so $|A|$ can never equal a negative number. Stop immediately.

When the Absolute Value Is Not Alone

If the equation looks like $3|x-2|-6 = 0$, the absolute value is not isolated. Before splitting, use algebra to get the absolute value by itself on one side:

$$3|x-2| - 6 = 0 \implies 3|x-2| = 6 \implies |x-2| = 2$$

Now apply the split rule.

The Four Steps

Step 1: Isolate the absolute value expression on one side of the equation.

Step 2: Check the right-hand side. If it is negative, write "no solution" and stop.

Step 3: Split into two equations: one with $+c$ and one with $-c$.

Step 4: Solve each equation and check both answers in the original equation.

Always Check Your Answers

Substituting back into the original equation catches errors, especially sign mistakes. This step is especially important when the expression inside the bars is more complex.

Pro Tip

- Before anything else, look at the right-hand side. If it is negative, the answer is immediately "no solution." Do not waste time splitting or solving.
- Isolate the absolute value *completely* before splitting. A common mistake is to split too early, before all the constants outside the bars have been moved.
- When writing the two cases, always make the entire expression inside the bars equal to both $+c$ *and* $-c$. Do not put the negative sign inside the bars.
- After solving, check each answer individually. Both answers should satisfy the original equation, not just the split equations.
- If $c = 0$, there is only one case to solve: $A = 0$. There is no second case because $+0$ and -0 are the same number.

Examples

Example 1 (Direct Split): Solve $|x| = 7$.

Step 1: The absolute value is already isolated.

Step 2: Check the right-hand side: $7 > 0$, so there are two solutions.

Step 3: Split.

$$x = 7 \qquad \text{or} \qquad x = -7$$

Step 4: Check.

$$|7| = 7 \quad \checkmark \qquad |-7| = 7 \quad \checkmark$$

$$\boxed{x = 7 \quad \text{or} \quad x = -7}$$

Example 2 (Expression Inside): Solve $|x - 3| = 5$.

Step 1: The absolute value is already isolated.

Step 2: Right-hand side is $5 > 0$. Two solutions.

Step 3: Split into two equations.

$$x - 3 = 5 \qquad \text{or} \qquad x - 3 = -5$$

Step 4: Solve each.

$$x - 3 = 5 \implies x = 8$$

$$x - 3 = -5 \implies x = -2$$

Check.

$$|8 - 3| = |5| = 5 \quad \checkmark \qquad |-2-3| = |-5| = 5 \quad \checkmark$$

$$\boxed{x = 8 \quad \text{or} \quad x = -2}$$

Example 3 (Linear Expression): Solve $|2x + 1| = 9$.

Step 1: The absolute value is already isolated.

Step 2: Right-hand side is $9 > 0$. Two solutions.

Step 3: Split.

$$2x + 1 = 9 \qquad \text{or} \qquad 2x + 1 = -9$$

Step 4: Solve each.

Case 1:

$$2x + 1 = 9$$
$$2x = 8$$
$$x = 4$$

Case 2:

$$2x + 1 = -9$$
$$2x = -10$$
$$x = -5$$

Check.

$$|2(4) + 1| = |9| = 9 \quad \checkmark \qquad |2(-5) + 1| = |-9| = 9 \quad \checkmark$$

$$\boxed{x = 4 \quad \text{or} \quad x = -5}$$

Example 4 (No Solution): Solve $|x| = -4$.

Step 1: The absolute value is already isolated.

Step 2: Check the right-hand side: $-4 < 0$.

The absolute value of any number is always greater than or equal to zero. It can never equal -4. There is no need to split.

$$\boxed{\text{No solution}}$$

Example 5 (Isolate First): Solve $3|x - 2| - 6 = 0$.

Step 1: Isolate the absolute value. Add 6 to both sides, then divide by 3.

$$3|x-2|-6=0$$

$$3|x-2|=6$$

$$|x-2|=2$$

Step 2: Right-hand side is $2>0$. Two solutions.

Step 3: Split.

$$x-2=2 \qquad \text{or} \qquad x-2=-2$$

Step 4: Solve each.

$$x-2=2 \implies x=4$$

$$x-2=-2 \implies x=0$$

Check in the original equation $3|x-2|-6=0$.

$$x=4: \quad 3|4-2|-6=3(2)-6=0 \quad \checkmark$$

$$x=0: \quad 3|0-2|-6=3(2)-6=0 \quad \checkmark$$

$$\boxed{x=4 \quad \text{or} \quad x=0}$$

Common Mistakes

- **Splitting when the right-hand side is negative:** If $|A|=-5$, the answer is immediately no solution. Students sometimes write $A=-5$ or $A=5$ anyway, which produces wrong answers.
- **Forgetting the negative case:** $|x-3|=5$ has two solutions. Solving only $x-3=5$ and stopping gives an incomplete answer. You must always write both cases.
- **Putting the negative inside the bars:** The two cases are $A=c$ and $A=-c$. Do not write $|-A|=c$ — the negative sign goes on the right-hand side, not inside the bars.
- **Splitting before isolating:** If the equation is $2|x|+3=9$, you must divide to get $|x|=3$ *before* splitting. Splitting $2|x|+3=9$ directly leads to incorrect cases.
- **Not checking answers:** Always substitute back into the original equation. Arithmetic errors in the split step can produce wrong values, and checking catches them.

Worksheet

Solve each equation. Show the split step clearly. Check both answers.

1. $|x| = 6$

2. $|x + 4| = 3$

3. $|x - 1| = 7$

4. $|2x - 3| = 5$

5. $|3x + 6| = 9$

6. $|x| = -2$

7. $|x - 5| = 0$

8. $|4x + 2| = 10$

9. $|2x - 1| = 0$

10. *(Challenge)* $|5x - 3| = 12$

11. *(Challenge)* $2|x + 1| + 4 = 10$

12. *(Challenge)* $-3|x - 2| + 9 = 0$

Answer Key

1. $|x| = 6$

Right-hand side $= 6 > 0$. Split:

$$x = 6 \qquad \text{or} \qquad x = -6$$

Check: $|6| = 6$ ✓ $\quad |-6| = 6$ ✓.

$$\boxed{x = 6 \quad \text{or} \quad x = -6}$$

2. $|x + 4| = 3$

Split:

$$x + 4 = 3 \implies x = -1 \qquad \text{or} \qquad x + 4 = -3 \implies x = -7$$

Check: $|-1 + 4| = |3| = 3$ ✓ $\quad |-7 + 4| = |-3| = 3$ ✓.

$$\boxed{x = -1 \quad \text{or} \quad x = -7}$$

3. $|x - 1| = 7$

Split:

$$x - 1 = 7 \implies x = 8 \qquad \text{or} \qquad x - 1 = -7 \implies x = -6$$

Check: $|8 - 1| = 7$ ✓ $\quad |-6 - 1| = |-7| = 7$ ✓.

$$\boxed{x = 8 \quad \text{or} \quad x = -6}$$

4. $|2x - 3| = 5$

Split:

$$2x - 3 = 5 \implies 2x = 8 \implies x = 4$$

$$2x - 3 = -5 \implies 2x = -2 \implies x = -1$$

Check: $|2(4) - 3| = |5| = 5$ ✓ $|2(-1) - 3| = |-5| = 5$ ✓.

$$\boxed{x = 4 \quad \text{or} \quad x = -1}$$

5. $|3x + 6| = 9$

Split:

$$3x + 6 = 9 \implies 3x = 3 \implies x = 1$$

$$3x + 6 = -9 \implies 3x = -15 \implies x = -5$$

Check: $|3(1) + 6| = |9| = 9$ ✓ $|3(-5) + 6| = |-9| = 9$ ✓.

$$\boxed{x = 1 \quad \text{or} \quad x = -5}$$

6. $|x| = -2$

Right-hand side $= -2 < 0$. Absolute value is never negative.

$$\boxed{\text{No solution}}$$

7. $|x - 5| = 0$

Right-hand side $= 0$. Only one case: $x - 5 = 0$.

$$x - 5 = 0 \implies x = 5$$

Check: $|5 - 5| = |0| = 0$ ✓.

$$\boxed{x = 5}$$

8. $|4x + 2| = 10$

Split:

$$4x + 2 = 10 \implies 4x = 8 \implies x = 2$$

$$4x + 2 = -10 \implies 4x = -12 \implies x = -3$$

Check: $|4(2) + 2| = |10| = 10$ ✓ $|4(-3) + 2| = |-10| = 10$ ✓.

$$\boxed{x = 2 \quad \text{or} \quad x = -3}$$

9. $|2x - 1| = 0$

Right-hand side $= 0$. Only one case: $2x - 1 = 0$.

$$2x = 1 \implies x = \frac{1}{2}$$

Check: $\left|2\left(\frac{1}{2}\right) - 1\right| = |0| = 0$ ✓.

$$\boxed{x = \frac{1}{2}}$$

10. *(Challenge)* $|5x - 3| = 12$

Split:

$$5x - 3 = 12 \implies 5x = 15 \implies x = 3$$
$$5x - 3 = -12 \implies 5x = -9 \implies x = -\frac{9}{5}$$

Check: $|5(3) - 3| = |12| = 12$ ✓ $\left|5\left(-\frac{9}{5}\right) - 3\right| = |-9-3| = |-12| = 12$ ✓.

$$\boxed{x = 3 \quad \text{or} \quad x = -\frac{9}{5}}$$

11. *(Challenge)* $2|x + 1| + 4 = 10$

Step 1: Isolate the absolute value.

$$2|x + 1| + 4 = 10$$
$$2|x + 1| = 6$$
$$|x + 1| = 3$$

Step 2: Right-hand side $= 3 > 0$. Split.

$$x + 1 = 3 \implies x = 2$$
$$x + 1 = -3 \implies x = -4$$

Check in original: $2|2 + 1| + 4 = 2(3) + 4 = 10$ ✓ $2|-4 + 1| + 4 = 2(3) + 4 = 10$ ✓.

$$\boxed{x = 2 \quad \text{or} \quad x = -4}$$

12. *(Challenge)* $-3|x - 2| + 9 = 0$

Step 1: Isolate the absolute value.

$$-3|x - 2| + 9 = 0$$
$$-3|x - 2| = -9$$
$$|x - 2| = 3$$

Step 2: Right-hand side $= 3 > 0$. Split.

$$x - 2 = 3 \implies x = 5$$

$$x - 2 = -3 \implies x = -1$$

Check in original: $-3|5-2| + 9 = -3(3) + 9 = 0$ ✓ $\quad -3|-1-2| + 9 = -3(3) + 9 = 0$ ✓.

$$\boxed{x = 5 \quad \text{or} \quad x = -1}$$

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