Solving Absolute Value Inequalities
ApexMath

Notes

The Two Types

Every absolute value inequality falls into one of two types, and each type has its own solution pattern. Understanding *why* each rule works — not just memorising it — is what makes these problems reliable.

Type 1: Less Than ($<$ or $\leq$) — The "AND" Case

When we write $|x| < 4$, we are asking: which values of x are *less than 4 units* from zero? The answer is all the values that sit within a distance of 4 from zero — a bounded interval in the middle.

$$|A| < c \qquad \Longrightarrow \qquad -c < A < c$$

This is a **compound AND inequality**: A must be greater than $-c$ *and* less than c at the same time. The solution is one connected interval.

$$|A| \leq c \qquad \Longrightarrow \qquad -c \leq A \leq c$$

Type 2: Greater Than ($>$ or $\geq$) — The "OR" Case

When we write $|x| > 5$, we are asking: which values of x are *more than 5 units* from zero? The answer is everything to the left of -5 or to the right of 5 — two separate rays going outward.

$$|A| > c \qquad \Longrightarrow \qquad A < -c \quad \text{or} \quad A > c$$

This is a **compound OR inequality**: A must satisfy one condition or the other. The solution is two disconnected rays.

$$|A| \geq c \qquad \Longrightarrow \qquad A \leq -c \quad \text{or} \quad A \geq c$$

Quick Memory Aid

- **Less than $\Rightarrow$ between**: one connected interval.
- **Greater than $\Rightarrow$ outside**: two separate rays.

Special Cases

- $|A| < 0$: No solution. Absolute value is never negative, so it cannot be less than zero.
- $|A| \geq 0$: All real numbers. Absolute value is always ≥ 0, so every x satisfies this.

- $|A| \le 0$: Only $A = 0$ works (absolute value equals exactly zero). One solution.

When the Absolute Value Is Not Isolated

Isolate the absolute value before applying either rule, exactly as in Lesson 3.

Writing the Answer

Solutions can be written three ways — all are equivalent and equally correct:

- **Inequality notation:** $-3 < x < 4$ or $x < -2$ or $x > 6$
- **Interval notation:** $(-3, 4)$ or $(-\infty, -2) \cup (6, \infty)$
- **Number line:** a shaded segment or two shaded rays

Pro Tip

- Before applying any rule, check whether the right-hand side is negative. If $|A| <$ negative number, stop and write no solution. If $|A| >$ negative number, stop and write all real numbers.
- The direction of the inequality symbol tells you the shape of the solution: $<$ or $\le$ gives a *segment* (bounded); $>$ or $\ge$ gives two *rays* (unbounded).
- After solving, always test one point from the solution region and one point outside it. If the inside point satisfies the inequality and the outside point does not, your answer is correct.
- Open dots ($\circ$) on the number line go with strict inequalities ($<$, $>$). Closed dots ($\bullet$) go with non-strict inequalities ($\le$, $\ge$).
- Isolate the absolute value completely before applying the rule. If you see $3|x+1| \le 12$, divide first to get $|x+1| \le 4$, then apply the rule.

Examples

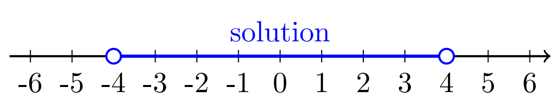
Example 1 (Less Than, Simple): Solve $|x| < 4$ and graph the solution.

Step 1: Identify the type. Less-than means AND (between).

Step 2: Apply the rule.

$$|x| < 4 \qquad \Longrightarrow \qquad -4 < x < 4$$

Step 3: Graph. Open dots at -4 and 4; shade the segment between them.



Check: Test $x = 0$ (inside): $|0| = 0 < 4$ ✓. Test $x = 6$ (outside): $|6| = 6 \not< 4$ ✓.

$$\boxed{-4 < x < 4}$$

Example 2 (Less Than or Equal, With Expression): Solve $|x-2| \leq 3$ and graph.

Step 1: Type. Less-than-or-equal means AND (between), with closed dots.

Step 2: Apply the rule.

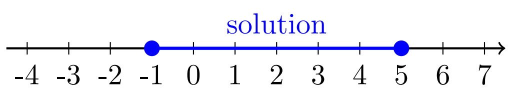
$$|x-2| \leq 3 \qquad \Longrightarrow \qquad -3 \leq x-2 \leq 3$$

Step 3: Solve the compound inequality.

Add 2 to all three parts:

$$-3+2 \leq x \leq 3+2$$
$$-1 \leq x \leq 5$$

Step 4: Graph. Closed dots at -1 and 5.



Check: Test $x=2$: $|2-2| = 0 \leq 3$ ✓. Test $x=7$: $|7-2| = 5 \not\leq 3$ ✓.

$$\boxed{-1 \leq x \leq 5}$$

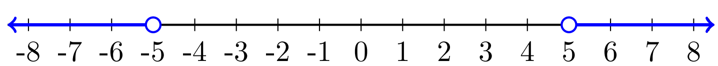
Example 3 (Greater Than, Simple): Solve $|x| > 5$ and graph.

Step 1: Type. Greater-than means OR (outside rays), open dots.

Step 2: Apply the rule.

$$|x| > 5 \qquad \Longrightarrow \qquad x < -5 \quad \text{or} \quad x > 5$$

Step 3: Graph. Open dots at -5 and 5; shade both outward rays.



Check: Test $x=0$ (between): $|0| = 0 \not> 5$ ✓. Test $x=-7$ (left ray): $|-7| = 7 > 5$ ✓.

$$\boxed{x < -5 \quad \text{or} \quad x > 5}$$

Example 4 (Less Than, Linear Expression): Solve $|2x-1| < 7$.

Step 1: Type. Less-than means AND (between).

Step 2: Apply the rule.

$$|2x-1| < 7 \qquad \Longrightarrow \qquad -7 < 2x-1 < 7$$

Step 3: Solve the compound inequality.

Add 1 to all three parts:

$$-7 + 1 < 2x < 7 + 1$$
$$-6 < 2x < 8$$

Divide all parts by 2:

$$-3 < x < 4$$

Check: Test $x = 0$: $|2(0) - 1| = 1 < 7$ ✓. Test $x = 5$: $|2(5) - 1| = 9 \not< 7$ ✓.

$$\boxed{-3 < x < 4}$$

Example 5 (Greater Than or Equal, Linear Expression): Solve $|x+3| \geq 2$ and graph.

Step 1: Type. Greater-than-or-equal means OR (outside), closed dots.

Step 2: Apply the rule.

$$|x + 3| \geq 2 \quad \Longrightarrow \quad x + 3 \leq -2 \quad \text{or} \quad x + 3 \geq 2$$

Step 3: Solve each inequality.

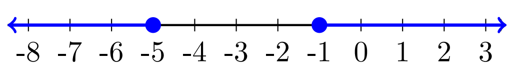
Left side:

$$x + 3 \leq -2 \implies x \leq -5$$

Right side:

$$x + 3 \geq 2 \implies x \geq -1$$

Step 4: Graph. Closed dots at -5 and -1; shade outward rays.



Check: Test $x = -3$ (between): $|-3+3| = 0 \not\geq 2$ ✓. Test $x = -6$ (left ray): $|-6+3| = 3 \geq 2$ ✓.

$$\boxed{x \leq -5 \quad \text{or} \quad x \geq -1}$$

Common Mistakes

- **Using the wrong rule for the wrong type:** Less-than gives a segment (AND); greater-than gives two rays (OR). Applying the greater-than rule to a less-than problem — or vice versa — gives the exact wrong region.
- **Forgetting to flip the sign on the left side:** $|x| < 4$ becomes $-4 < x < 4$. The left boundary is $-c$, not just c. Writing $0 < x < 4$ or $4 < x < 4$ are both wrong.

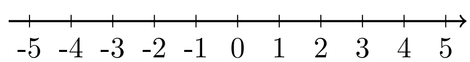
- **Writing AND instead of OR for greater-than:** $|x| > 5$ gives $x < -5$ *or* $x > 5$ — two separate pieces. Writing $-5 < x > 5$ is not valid inequality notation and describes the wrong region.
- **Using open dots for $\leq$ or $\geq$:** Closed dots mean the endpoint is included. If the inequality is non-strict ($\leq$ or $\geq$), use a filled dot. Only use an open dot for strict inequalities ($<$ or $>$).
- **Not isolating before applying the rule:** For $3|x+1| \leq 12$, you must divide first to get $|x+1| \leq 4$, then write $-4 \leq x+1 \leq 4$. Applying the rule to $3|x+1|$ directly gives wrong boundaries.

Worksheet

Solve each inequality. Write the solution in inequality notation and draw the solution on the number line. Label the boundary points.

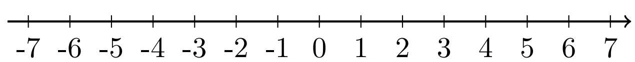
1. $|x| < 3$

   Solution: ____________________________



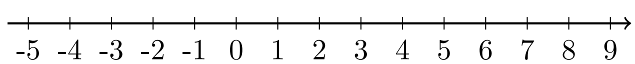
2. $|x| > 4$

   Solution: ____________________________



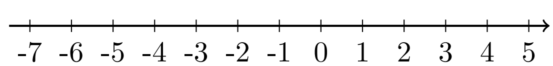
3. $|x-2| \leq 5$

   Solution: ____________________________



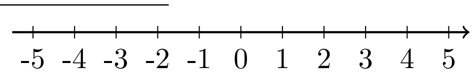
4. $|x+1| \geq 3$

   Solution: ____________________________



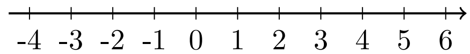
5. $|2x| < 6$

   Solution: ____________________________



6. $|3x - 3| > 6$

Solution: ______________________



7. $|x - 4| < 0$

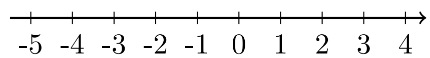
Solution: ______________________

8. $|x + 2| \geq 0$

Solution: ______________________

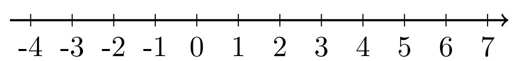
9. *(Challenge)* $|2x + 1| \leq 5$

Solution: ______________________



10. *(Challenge)* $|3x - 6| > 9$

Solution: ______________________

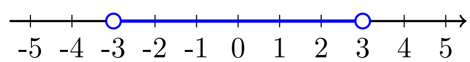


Answer Key

1. $|x| < 3$

Type: less-than (AND, between). Apply rule:

$$-3 < x < 3$$

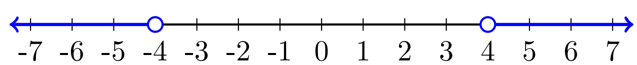


$$\boxed{-3 < x < 3}$$

2. $|x| > 4$

Type: greater-than (OR, outside). Apply rule:

$$x < -4 \quad \text{or} \quad x > 4$$



$$\boxed{x < -4 \quad \text{or} \quad x > 4}$$

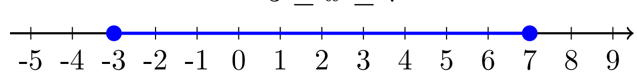
3. $|x - 2| \leq 5$

Type: less-than-or-equal (AND, between, closed dots).

$$-5 \leq x - 2 \leq 5$$

Add 2 to all parts:

$$-3 \leq x \leq 7$$



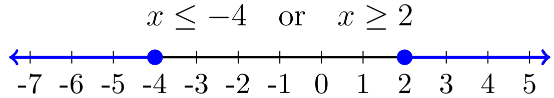
$$\boxed{-3 \leq x \leq 7}$$

4. $|x + 1| \geq 3$

Type: greater-than-or-equal (OR, outside, closed dots).

$$x + 1 \leq -3 \quad \text{or} \quad x + 1 \geq 3$$

$$x \leq -4 \quad \text{or} \quad x \geq 2$$



$$\boxed{x \leq -4 \quad \text{or} \quad x \geq 2}$$

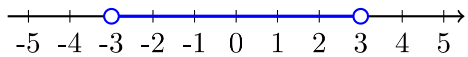
5. $|2x| < 6$

Type: less-than (AND, between).

$$-6 < 2x < 6$$

Divide all parts by 2:

$$-3 < x < 3$$



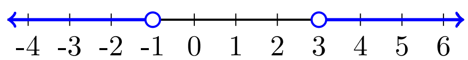
$$\boxed{-3 < x < 3}$$

6. $|3x - 3| > 6$

Type: greater-than (OR, outside).

$$3x - 3 < -6 \quad \text{or} \quad 3x - 3 > 6$$

Left: $3x < -3 \Rightarrow x < -1$. Right: $3x > 9 \Rightarrow x > 3$.



$$\boxed{x < -1 \quad \text{or} \quad x > 3}$$

7. $|x - 4| < 0$

Absolute value is always ≥ 0. It can never be strictly less than 0.

$$\boxed{\text{No solution}}$$

8. $|x + 2| \geq 0$

Absolute value is always ≥ 0 for every real number x.

$$\boxed{\text{All real numbers}}$$

9. *(Challenge)* $|2x + 1| \leq 5$

Type: less-than-or-equal (AND, between, closed dots).

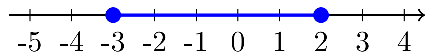
$$-5 \leq 2x + 1 \leq 5$$

Subtract 1 from all parts:

$$-6 \leq 2x \leq 4$$

Divide all parts by 2:

$$-3 \leq x \leq 2$$



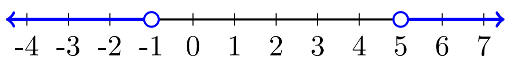
$$\boxed{-3 \le x \le 2}$$

10. *(Challenge)* $|3x - 6| > 9$

Type: greater-than (OR, outside).

$$3x - 6 < -9 \quad \text{or} \quad 3x - 6 > 9$$

Left: $3x < -3 \Rightarrow x < -1$. Right: $3x > 15 \Rightarrow x > 5$.



$$\boxed{x < -1 \quad \text{or} \quad x > 5}$$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson · ApexMath
