Transformations of $y = |x|$
ApexMath

Notes

The General Form

Every transformed absolute value function can be written as:

$$\boxed{y = a\,|x - h| + k}$$

Each parameter controls a specific feature of the graph:

- h shifts the graph **horizontally**. The vertex moves to $x = h$. Note: $|x - h|$ shifts *right* when $h > 0$ and *left* when $h < 0$.
- k shifts the graph **vertically**. The vertex moves to $y = k$. Positive k shifts up; negative k shifts down.
- a controls the **direction** and **steepness**:
  - If $a > 0$, the V opens **upward**.
  - If $a < 0$, the V opens **downward** (reflected).
  - $|a| > 1$ makes the arms **steeper** (vertical stretch).
  - $|a| < 1$ makes the arms **shallower** (vertical compression).
  - The slope of the right arm is $+a$; the slope of the left arm is $-a$.

The Vertex

The vertex of $y = a|x - h| + k$ is always at the point $(\mathbf{h}, \mathbf{k})$.

This is the most important thing to read from the equation. Once you know the vertex, you can sketch the full graph in seconds by going one unit to the right and $|a|$ units up (or down if $a < 0$) to find the next point.

Domain and Range

- **Domain:** always all real numbers, $(-\infty, \infty)$, regardless of the transformation.
- **Range:** depends on the direction.
  - If $a > 0$ (opens up): range is $y \geq k$, written $[k, \infty)$.
  - If $a < 0$ (opens down): range is $y \leq k$, written $(-\infty, k]$.

Pro Tip

- Read the vertex directly from the equation before doing anything else. For $y = 2|x - 3| + 1$, the vertex is $(3, 1)$ — not $(-3, 1)$ and not $(3, 0)$. The sign inside the bars is opposite to the h-value.

- The fastest way to sketch: plot the vertex, then go one unit right and $|a|$ units up/down, and one unit left and $|a|$ units up/down. Three points are enough to draw the V.
- If $a < 0$, the V is upside down. The vertex is now the *highest* point, not the lowest.
- Watch for $|x + h|$. Since $|x + h| = |x - (-h)|$, the horizontal shift is $-h$: for example, $|x + 3|$ has vertex at $x = -3$, not $x = 3$.
- To write the equation from a graph: read the vertex (h, k), determine the direction (up or down sets the sign of a), then find $|a|$ by counting how many units up/down the graph goes for every one unit across.

Examples

Example 1 (Vertical Shift): Sketch $y = |x| + 3$ and state the vertex, domain, and range.

Step 1: Identify the parameters.

Comparing to $y = a|x - h| + k$: $a = 1$, $h = 0$, $k = 3$.

Step 2: Locate the vertex.

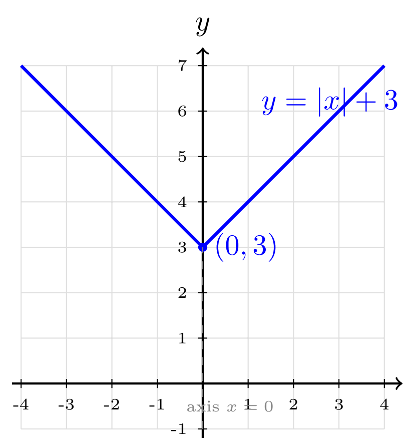
$$\text{Vertex} = (h, k) = (0,\ 3)$$

The parent graph $y = |x|$ has been shifted **3 units up**.

Step 3: Find two more points. Go 1 right, 1 up (slope $+1$); 1 left, 1 up (slope -1).

$$(1,\ 4) \qquad (-1,\ 4)$$

Step 4: State features. Domain: $(-\infty, \infty)$. Range: $y \geq 3$, i.e. $[3, \infty)$.



Example 2 (Horizontal Shift): Sketch $y = |x - 2|$ and state the vertex, domain, and range.

Step 1: Identify parameters.

$a = 1$, $h = 2$, $k = 0$.

Step 2: Vertex.

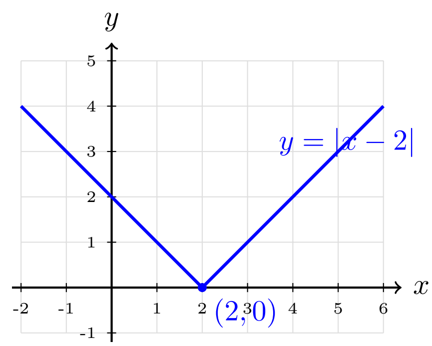
$$\text{Vertex} = (2,\ 0)$$

The graph has shifted **2 units right**. Note: $|x - 2|$ gives $h = +2$ (shift right), not left.

Step 3: Two more points.

$$(3,\ 1) \qquad (1,\ 1)$$

Step 4: Features. Domain: $(-\infty, \infty)$. Range: $y \geq 0$, i.e. $[0, \infty)$.



Example 3 (Reflection): Sketch $y = -|x|$ and state the vertex, domain, and range.

Step 1: Identify parameters.

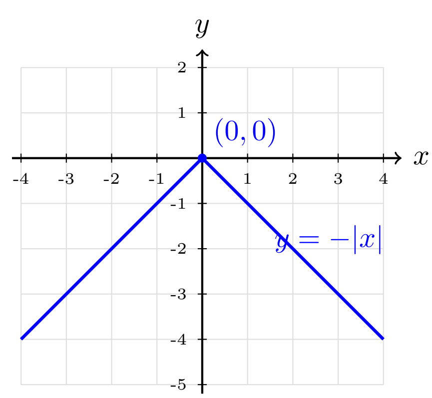
$a = -1$, $h = 0$, $k = 0$.

Step 2: Vertex.

$$\text{Vertex} = (0,\ 0)$$

Step 3: Direction. Since $a = -1 < 0$, the V opens **downward**. The slopes are -1 (right arm going down) and $+1$ (left arm going down to the right). Two points: $(1, -1)$ and $(-1, -1)$.

Step 4: Features. Domain: $(-\infty, \infty)$. Range: $y \leq 0$, i.e. $(-\infty, 0]$.



Example 4 (Vertical Stretch): Sketch $y = 2|x|$ and compare it to $y = |x|$.

Step 1: Identify parameters.

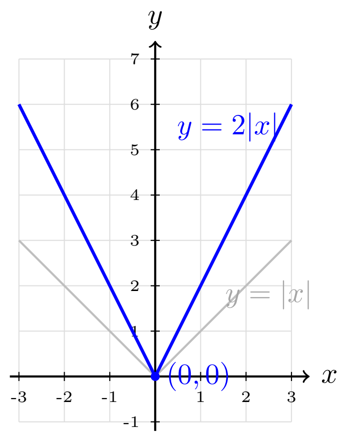
$a = 2$, $h = 0$, $k = 0$.

Step 2: Vertex.

$$\text{Vertex} = (0,\ 0)$$

Step 3: Direction and steepness. Since $a = 2 > 1$, the arms are **steeper** than the parent. For every 1 unit across, the graph rises 2 units (slope ± 2). Two points: $(1, 2)$ and $(-1, 2)$.

Step 4: Features. Domain: $(-\infty, \infty)$. Range: $y \geq 0$, i.e. $[0, \infty)$.



The grey graph is the parent $y = |x|$ for comparison. The blue $y = 2|x|$ is steeper.

Example 5 (Combined Transformation): Sketch $y = -2|x-1|+4$ and state all features.

Step 1: Identify parameters.

$a = -2$, $h = 1$, $k = 4$.

Step 2: Vertex.

$$\text{Vertex} = (1,\ 4)$$

Step 3: Direction and slope. $a = -2 < 0$, so the V opens **downward**. The arms have slope magnitude 2: for every 1 unit across, the graph drops 2 units. Two more points:

$$x = 0: \quad y = -2|0-1| + 4 = -2(1) + 4 = 2$$

$$x = 2: \quad y = -2|2-1| + 4 = -2(1) + 4 = 2$$

And further out:

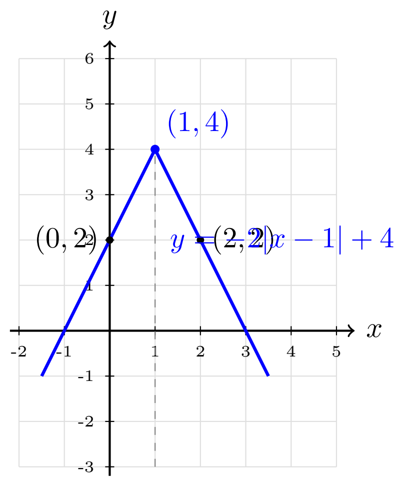
$$x = -1: \quad y = -2|-1-1| + 4 = -2(2) + 4 = 0$$

$$x = 3: \quad y = -2|3-1| + 4 = -2(2) + 4 = 0$$

Step 4: Features.

- Vertex: $(1, 4)$ — the highest point
- Axis of symmetry: $x = 1$

- Domain: $(-\infty, \infty)$
- Range: $y \le 4$, i.e. $(-\infty, 4]$



Common Mistakes

- **Reversing the horizontal shift:** In $y = |x - 3|$, the vertex is at $x = 3$, not $x = -3$. The shift is *opposite* to the sign inside the bars. Think of it as: "what value of x makes the inside equal zero?" For $|x - 3|$, that is $x = 3$.
- **Forgetting the negative in $y = a|x - h| + k$ when $a < 0$:** When a is negative, the V flips downward. The vertex becomes the *maximum*, not the minimum. The range is $(-\infty, k]$, not $[k, \infty)$.
- **Confusing slope with a:** The right arm has slope $+a$ and the left arm has slope $-a$. If $a = 2$, the right arm has slope $+2$ and the left arm has slope -2. If $a = -2$, the right arm has slope -2 and the left arm has slope $+2$.
- **Not adjusting the range when $k \neq 0$:** The range of $y = |x| + 3$ is $y \ge 3$, not $y \ge 0$. The vertex moves to $(0, 3)$, so the minimum output is 3.
- **Plotting the wrong vertex:** Always identify (h, k) first and mark it before drawing anything else. Every other point follows from the vertex.

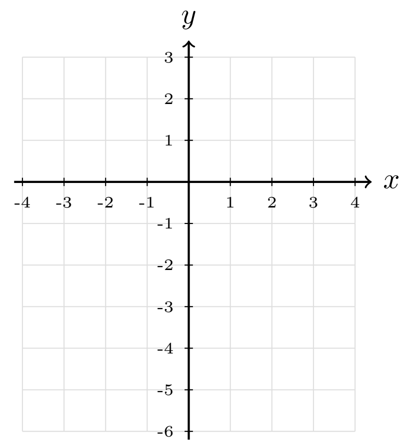
Worksheet

Part 1 — For each equation, state the vertex, opening direction, slope magnitude, domain, and range. Then sketch the graph.

1. $y = |x| - 4$

   Vertex: __________ Direction: __________ Slope magnitude: ________

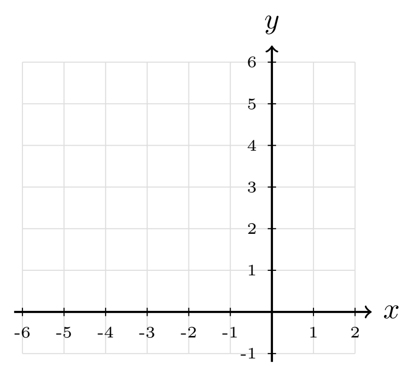
   Domain: _______________ Range: _______________



2. $y = |x + 3|$

   Vertex: ________ Direction: ________ Slope magnitude: ______

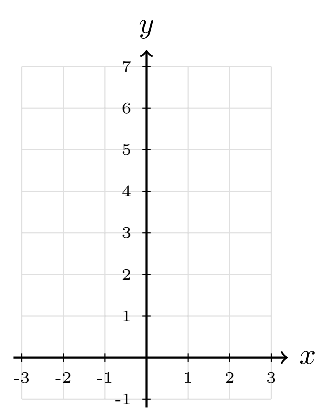
   Domain: ____________ Range: ____________



3. $y = 3|x|$

   Vertex: ________ Direction: ________ Slope magnitude: ______

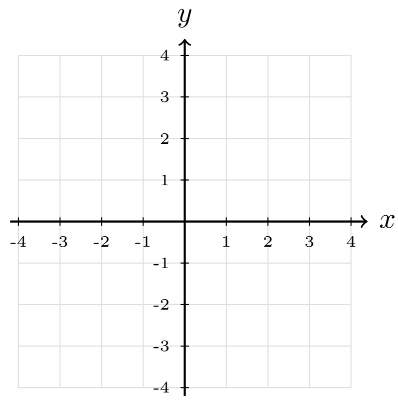
   Domain: ____________ Range: ____________



4. $y = -|x| + 2$

Vertex: __________ Direction: __________ Slope magnitude: ________

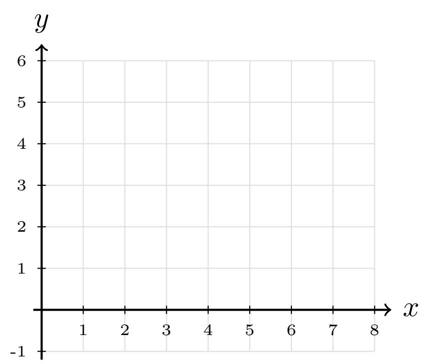
Domain: ________________ Range: ________________



5. $y = |x - 4| + 1$

Vertex: __________ Direction: __________ Slope magnitude: ________

Domain: ________________ Range: ________________

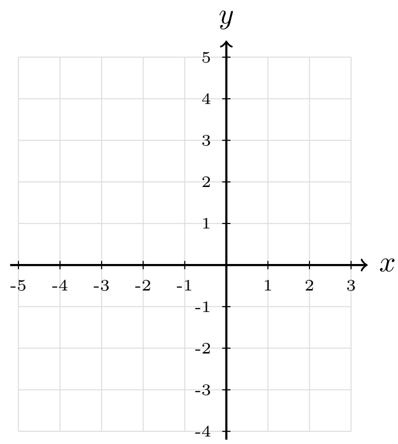


Part 2 — For each equation, state the vertex, opening direction, slope magnitude, domain, and range. Sketch the graph.

6. $y = 2|x + 1| - 3$

Vertex: __________ Direction: __________ Slope magnitude: ________

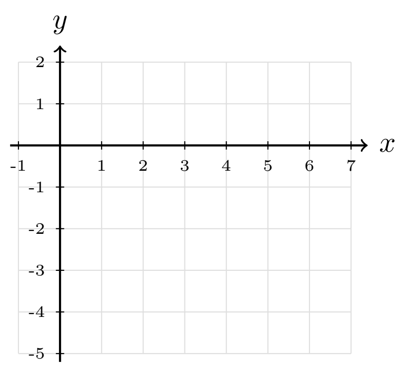
Domain: ________________ Range: ________________



7. $y = -|x - 3|$

   Vertex: __________ Direction: __________ Slope magnitude: ________

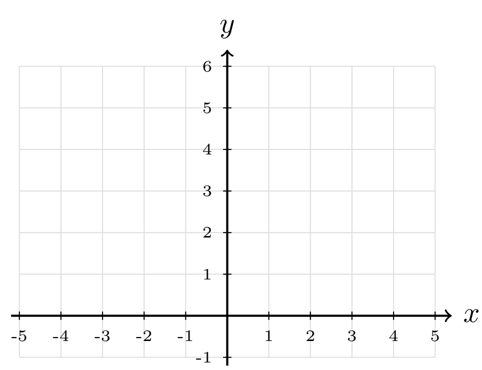
   Domain: _______________ Range: _______________



8. $y = \frac{1}{2}|x| + 2$

   Vertex: __________ Direction: __________ Slope magnitude: ________

   Domain: _______________ Range: _______________



Part 3 — Challenge: Write the equation of each function from its description.

9. Vertex at $(-2, 5)$, opens downward, slope magnitude 3.

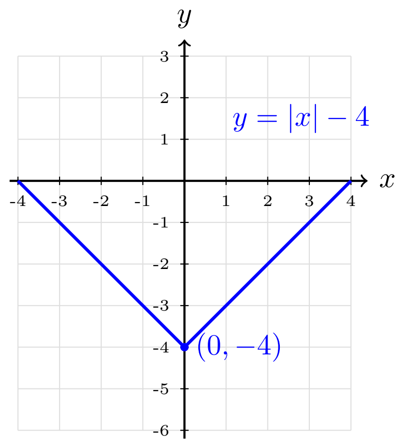
10. Vertex at $(1, -4)$, opens upward, slope magnitude 2.

Answer Key

1. $y = |x| - 4$: $\quad a = 1,\ h = 0,\ k = -4.$

Vertex $(0, -4)$, opens **up**, slope magnitude 1. Domain $(-\infty, \infty)$, Range $[-4, \infty)$.

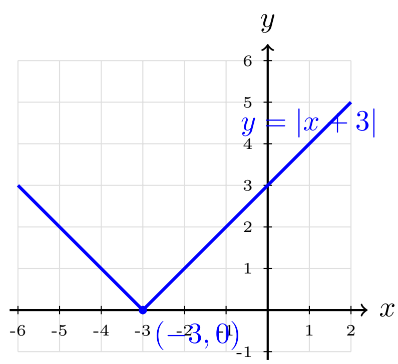
Key points: $(0, -4)$, $(1, -3)$, $(-1, -3)$, $(2, -2)$, $(-2, -2)$.



2. $y = |x + 3|$: $\quad a = 1,\ h = -3,\ k = 0.$

Vertex $(-3, 0)$, opens **up**, slope magnitude 1. Domain $(-\infty, \infty)$, Range $[0, \infty)$.

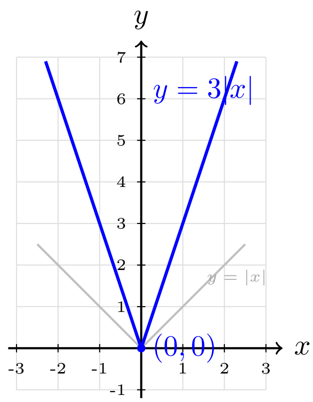
Key points: $(-3, 0)$, $(-2, 1)$, $(-4, 1)$, $(-1, 2)$, $(-5, 2)$.



3. $y = 3|x|$: $\quad a = 3,\ h = 0,\ k = 0.$

Vertex $(0, 0)$, opens **up**, slope magnitude 3. Domain $(-\infty, \infty)$, Range $[0, \infty)$.

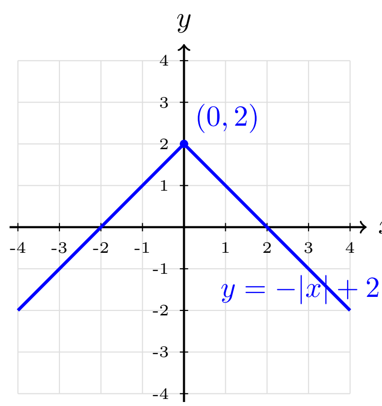
Key points: $(0, 0)$, $(1, 3)$, $(-1, 3)$, $(2, 6)$, $(-2, 6)$.



4. $y = -|x| + 2$: $\quad a = -1,\ h = 0,\ k = 2.$

Vertex $(0,2)$, opens **down**, slope magnitude 1. Domain $(-\infty,\infty)$, Range $(-\infty,2]$.

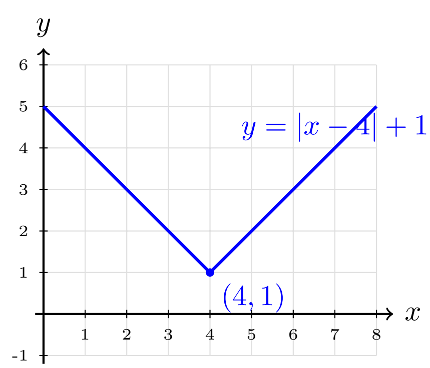
Key points: $(0,2)$, $(1,1)$, $(-1,1)$, $(2,0)$, $(-2,0)$.



5. $y = |x-4| + 1$: $\quad a = 1,\ h = 4,\ k = 1.$

Vertex $(4,1)$, opens **up**, slope magnitude 1. Domain $(-\infty,\infty)$, Range $[1,\infty)$.

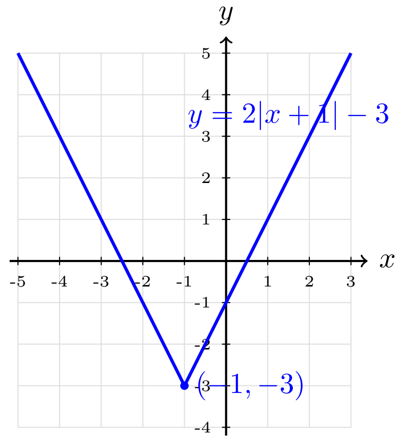
Key points: $(4,1)$, $(5,2)$, $(3,2)$, $(6,3)$, $(2,3)$.



6. $y = 2|x+1| - 3$: $\quad a = 2,\ h = -1,\ k = -3.$

Vertex $(-1,-3)$, opens **up**, slope magnitude 2. Domain $(-\infty,\infty)$, Range $[-3,\infty)$.

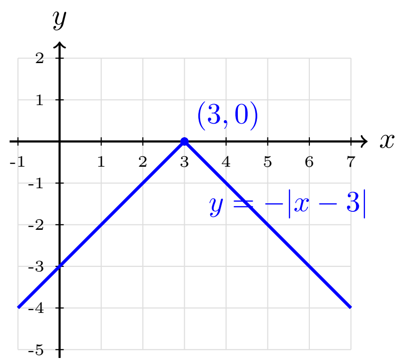
Key points: $(-1,-3)$, $(0,-1)$, $(-2,-1)$, $(1,1)$, $(-3,1)$.



7. $y = -|x-3|$: $\quad a = -1,\ h = 3,\ k = 0.$

Vertex $(3,0)$, opens **down**, slope magnitude 1. Domain $(-\infty,\infty)$, Range $(-\infty,0]$.

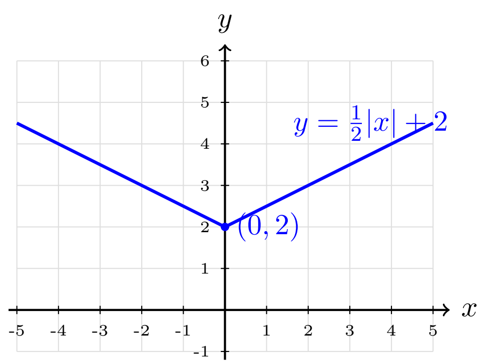
Key points: $(3,0)$, $(4,-1)$, $(2,-1)$, $(5,-2)$, $(1,-2)$.



8. $y = \frac{1}{2}|x| + 2$: $\quad a = \frac{1}{2},\ h = 0,\ k = 2.$

Vertex $(0,2)$, opens **up**, slope magnitude $\frac{1}{2}$. Domain $(-\infty,\infty)$, Range $[2,\infty)$.

Key points: $(0,2)$, $(2,3)$, $(-2,3)$, $(4,4)$, $(-4,4)$.



9. Vertex $(-2,5)$, opens down, slope magnitude 3.

The vertex gives $h = -2$, $k = 5$. Opens down means $a < 0$. Slope magnitude 3 means $|a| = 3$, so $a = -3$.

$$y = -3|x - (-2)| + 5 = -3|x + 2| + 5$$

Verify vertex: $y(-2) = -3|-2 + 2| + 5 = -3(0) + 5 = 5$ ✓.

$$\boxed{y = -3|x + 2| + 5}$$

10. Vertex $(1, -4)$, opens up, slope magnitude 2.

The vertex gives $h = 1$, $k = -4$. Opens up means $a > 0$. Slope magnitude 2 means $a = 2$.

$$y = 2|x - 1| + (-4) = 2|x - 1| - 4$$

Verify vertex: $y(1) = 2|1 - 1| - 4 = 0 - 4 = -4$ ✓.

Verify one more point: $x = -1$: $y = 2| - 1 - 1| - 4 = 2(2) - 4 = 0$ ✓.

$$\boxed{y = 2|x - 1| - 4}$$

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End of Lesson · ApexMath
