Arithmetic Sequences
ApexMath

Notes

What Is an Arithmetic Sequence?

An **arithmetic sequence** is a list of numbers where the difference between any two consecutive terms is always the same. That fixed difference is called the **common difference** and is written as d.

$$d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots$$

If the terms are increasing, then $d > 0$. If the terms are decreasing, then $d < 0$.

Examples of Arithmetic Sequences

- $3,\ 7,\ 11,\ 15,\ 19,\ \ldots$ Here $d = 4$ (add 4 each time)
- $20,\ 15,\ 10,\ 5,\ 0,\ \ldots$ Here $d = -5$ (subtract 5 each time)
- $1,\ 1.5,\ 2,\ 2.5,\ 3,\ \ldots$ Here $d = 0.5$

The General Term Formula

Instead of listing every term one by one, we use a formula. The nth term of an arithmetic sequence is:

$$\boxed{t_n = t_1 + (n-1)\,d}$$

Where:

- t_n is the term you want to find
- t_1 is the first term of the sequence
- d is the common difference
- n is the position number of the term

Why $(n-1)$ and not n?

Think of it this way. To get from t_1 to t_2, you add d once. To get from t_1 to t_3, you add d twice. To get from t_1 to t_n, you add d exactly $(n-1)$ times. You never add d to reach t_1 itself, so you start counting from zero — that is why the exponent is $(n-1)$, not n.

Finding t_1 and d When You Are Given Two Terms

Sometimes you are not given t_1 and d directly. Instead you are told two terms, such as $t_4 = 22$ and $t_9 = 47$. You can find t_1 and d by setting up two equations and solving the system.

Write the formula for each given term:

$$t_4 = t_1 + 3d = 22 \qquad \text{and} \qquad t_9 = t_1 + 8d = 47$$

Subtract the first equation from the second to eliminate t_1, solve for d, then substitute back to find t_1. This method will be shown in full in Example 5.

Is a Given Value a Term of the Sequence?

To test whether a value v is a term in the sequence, set $t_n = v$ and solve for n. If n turns out to be a positive whole number (a positive integer), then v is a term. If n is a fraction or a negative number, then v is not a term.

Pro Tip

- **Always find d first.** Before using any formula, subtract two consecutive terms to find d. Getting d wrong will make every answer that follows wrong.
- **Write out the formula with values substituted before simplifying.** Students who skip straight to arithmetic often make errors. Write $t_n = t_1 + (n-1)\,d$, then substitute, then simplify step by step.
- **When solving for n, check that your answer is a positive whole number.** If you get $n = 7.5$ or $n = -3$, the value is not a term in the sequence.
- **When given two non-consecutive terms, subtract the equations to eliminate t_1.** This is faster and cleaner than trying to guess d.
- **Negative d is fine.** A decreasing sequence is still arithmetic. Just be careful with signs when you substitute a negative d into the formula.

Examples

Example 1 (Listing terms from t_1 and d): Write the first five terms of the arithmetic sequence with $t_1 = 7$ and $d = 5$.

Step 1: Start with t_1.

$$t_1 = 7$$

Step 2: Add $d = 5$ repeatedly.

$$t_2 = 7 + 5 = 12$$
$$t_3 = 12 + 5 = 17$$
$$t_4 = 17 + 5 = 22$$
$$t_5 = 22 + 5 = 27$$

$$\boxed{7,\ 12,\ 17,\ 22,\ 27}$$

Example 2 (Decreasing sequence): Write the first five terms of the arithmetic sequence with $t_1 = 30$ and $d = -4$.

Step 1: Start with t_1.

$$t_1 = 30$$

Step 2: Add $d = -4$ each time (i.e. subtract 4).

$$t_2 = 30 + (-4) = 26$$
$$t_3 = 26 + (-4) = 22$$
$$t_4 = 22 + (-4) = 18$$
$$t_5 = 18 + (-4) = 14$$

$$\boxed{30,\ 26,\ 22,\ 18,\ 14}$$

Example 3 (Finding the general term and a specific term): The sequence is $3,\ 7,\ 11,\ 15,\ \ldots$ Find the general term t_n and use it to find t_{50}.

Step 1: Identify t_1 and d.

$$t_1 = 3 \qquad d = 7 - 3 = 4$$

Step 2: Write the general term formula and substitute.

$$t_n = t_1 + (n-1)\,d = 3 + (n-1)(4)$$

Step 3: Expand and simplify.

$$t_n = 3 + 4n - 4 = 4n - 1$$

Step 4: Find t_{50} by substituting $n = 50$.

$$t_{50} = 4(50) - 1 = 200 - 1$$

$$\boxed{t_n = 4n - 1, \qquad t_{50} = 199}$$

Example 4 (Solving for n): An arithmetic sequence has $t_1 = 40$ and $d = -6$. Find the general term, then find which term equals -20.

Step 1: Write the general term.

$$t_n = 40 + (n-1)(-6) = 40 - 6n + 6 = 46 - 6n$$

Step 2: Set $t_n = -20$ and solve for n.

$$46 - 6n = -20$$

Step 3: Isolate n.

$$-6n = -20 - 46$$

$$-6n = -66$$

$$n = \frac{-66}{-6} = 11$$

Step 4: Since $n = 11$ is a positive whole number, it is valid. Verify.

$$t_{11} = 46 - 6(11) = 46 - 66 = -20\checkmark$$

$$\boxed{t_n = 46 - 6n, \qquad -20 \text{ is the 11th term}}$$

Example 5 (Finding t_1 and d from two terms): An arithmetic sequence has $t_4 = 22$ and $t_9 = 47$. Find t_1, d, and t_{20}.

Step 1: Write the formula for each given term.
For t_4:

$$t_1 + 3d = 22 \qquad \cdots (1)$$

For t_9:

$$t_1 + 8d = 47 \qquad \cdots (2)$$

Step 2: Subtract equation (1) from equation (2) to eliminate t_1.

$$(t_1 + 8d) - (t_1 + 3d) = 47 - 22$$
$$5d = 25$$
$$d = 5$$

Step 3: Substitute $d = 5$ into equation (1) to find t_1.

$$t_1 + 3(5) = 22$$
$$t_1 + 15 = 22$$
$$t_1 = 7$$

Step 4: Find t_{20}.

$$t_{20} = 7 + (20 - 1)(5) = 7 + 19 \times 5 = 7 + 95 = 102$$

$$\boxed{t_1 = 7, \quad d = 5, \quad t_{20} = 102}$$

Example 6 (Is a value a term of the sequence?): The arithmetic sequence has $t_1 = 2$ and $d = 3$. Is 98 a term of this sequence?

Step 1: Write the general term.

$$t_n = 2 + (n - 1)(3) = 2 + 3n - 3 = 3n - 1$$

Step 2: Set $t_n = 98$ and solve for n.

$$3n - 1 = 98$$
$$3n = 99$$
$$n = 33$$

Step 3: Check that n is a positive whole number.
$n = 33$ is a positive integer, so yes, 98 is a term.
Step 4: Verify.

$$t_{33} = 3(33) - 1 = 99 - 1 = 98\checkmark$$

$$\boxed{\text{98 is the 33rd term of the sequence.}}$$

Common Mistakes

- **Writing $t_n = t_1 + nd$ instead of $t_n = t_1 + (n-1)d$.** The exponent is $(n-1)$, not n. Using n gives every term a value that is exactly d too large.
- **Computing d by dividing instead of subtracting.** Division is for geometric sequences. For arithmetic sequences, always subtract: $d = t_2 - t_1$.
- **Forgetting the negative sign on d.** If the sequence is decreasing, d is negative. Substituting a positive number in its place will give wrong answers.
- **Accepting a fractional or negative n as a valid answer.** When solving for n, if the result is not a positive whole number, the value is not a term in the sequence.
- **Setting up the equations wrong when given two terms.** Remember that $t_4 = t_1 + 3d$ (not $t_1 + 4d$), because you add d exactly $(n-1)$ times. Always subtract one from the position number.

Worksheet

1. The sequence $2,\ 6,\ 10,\ 14,\ \ldots$ is arithmetic.
   (a) State t_1 and d.
   (b) Write the general term t_n in simplified form.
   (c) Find t_{25}.

2. The sequence $50,\ 44,\ 38,\ 32,\ \ldots$ is arithmetic.
   (a) State t_1 and d.
   (b) Write the general term t_n in simplified form.
   (c) Find t_{15}.

3. An arithmetic sequence has $t_1 = 8$ and $d = 7$.

   (a) Write the general term t_n.

   (b) Find t_{20}.

4. An arithmetic sequence has $t_1 = 100$ and $d = -9$.

   (a) Write the general term t_n.

   (b) Find t_{12}.

   (c) Which is the first term of the sequence that is negative?

5. An arithmetic sequence has $t_4 = 13$ and $t_{10} = 37$. Find t_1, d, and t_{30}.

6. An arithmetic sequence has $t_3 = 5$ and $t_8 = -20$. Find t_1, d, and the general term t_n.

7. The sequence 5, 11, 17, 23, ... is arithmetic. Is 200 a term of this sequence? Show your work and justify your answer.

8. The sequence 4, 9, 14, 19, ... is arithmetic. Is 84 a term of this sequence? If so, which term is it?

9. *(Challenge)* Insert three arithmetic means between 8 and 28.
(Hint: You need five equally-spaced terms total, with 8 *as the first and* 28 *as the last. Find d first.)*

10. *(Challenge)* The first three terms of an arithmetic sequence are $x + 2$, $2x - 1$, and $4x - 3$. Find the value of x, then write the first three terms and find t_{10}.

Answer Key

1. Sequence: 2, 6, 10, 14, ...

(a) State t_1 and d.

$$t_1 = 2 \qquad d = 6 - 2 = 4$$

$$\boxed{t_1 = 2, \quad d = 4}$$

(b) General term.

Step 1: Write the formula and substitute.

$$t_n = t_1 + (n-1)\,d = 2 + (n-1)(4)$$

Step 2: Expand and simplify.

$$t_n = 2 + 4n - 4 = 4n - 2$$

$$\boxed{t_n = 4n - 2}$$

(c) Find t_{25}.

$$t_{25} = 4(25) - 2 = 100 - 2$$

$$\boxed{t_{25} = 98}$$

2. Sequence: 50, 44, 38, 32, ...
(a) State t_1 and d.

$$t_1 = 50 \qquad d = 44 - 50 = -6$$

$$\boxed{t_1 = 50, \quad d = -6}$$

(b) General term.
Step 1: Substitute into the formula.

$$t_n = 50 + (n-1)(-6)$$

Step 2: Expand and simplify.

$$t_n = 50 - 6n + 6 = 56 - 6n$$

$$\boxed{t_n = 56 - 6n}$$

(c) Find t_{15}.

$$t_{15} = 56 - 6(15) = 56 - 90$$

$$\boxed{t_{15} = -34}$$

3. $t_1 = 8$, $d = 7$
(a) General term.
Step 1: Substitute into the formula.

$$t_n = 8 + (n-1)(7)$$

Step 2: Expand and simplify.

$$t_n = 8 + 7n - 7 = 7n + 1$$

$$\boxed{t_n = 7n + 1}$$

(b) Find t_{20}.

$$t_{20} = 7(20) + 1 = 140 + 1$$

$$\boxed{t_{20} = 141}$$

4. $t_1 = 100$, $d = -9$

(a) General term.

Step 1: Substitute into the formula.

$$t_n = 100 + (n-1)(-9)$$

Step 2: Expand and simplify.

$$t_n = 100 - 9n + 9 = 109 - 9n$$

$$\boxed{t_n = 109 - 9n}$$

(b) Find t_{12}.

$$t_{12} = 109 - 9(12) = 109 - 108$$

$$\boxed{t_{12} = 1}$$

(c) First negative term.

Step 1: Set $t_n < 0$ and solve.

$$109 - 9n < 0$$
$$-9n < -109$$
$$n > \frac{109}{9} \approx 12.1$$

Step 2: The smallest whole number greater than 12.1 is $n = 13$.

$$t_{13} = 109 - 9(13) = 109 - 117 = -8$$

$$\boxed{\text{The first negative term is } t_{13} = -8}$$

5. $t_4 = 13$, $t_{10} = 37$

Step 1: Write an equation for each given term.

For t_4:

$$t_1 + 3d = 13 \qquad \cdots (1)$$

For t_{10}:

$$t_1 + 9d = 37 \qquad \cdots (2)$$

Step 2: Subtract equation (1) from equation (2).

$$(t_1 + 9d) - (t_1 + 3d) = 37 - 13$$

$$6d = 24$$

$$d = 4$$

Step 3: Substitute $d = 4$ into equation (1).

$$t_1 + 3(4) = 13$$

$$t_1 + 12 = 13$$

$$t_1 = 1$$

Step 4: Find t_{30}.

$$t_{30} = 1 + (30 - 1)(4) = 1 + 29 \times 4 = 1 + 116$$

$$\boxed{t_1 = 1, \quad d = 4, \quad t_{30} = 117}$$

6. $t_3 = 5$, $t_8 = -20$
Step 1: Write an equation for each given term.
For t_3:

$$t_1 + 2d = 5 \qquad \cdots (1)$$

For t_8:

$$t_1 + 7d = -20 \qquad \cdots (2)$$

Step 2: Subtract equation (1) from equation (2).

$$(t_1 + 7d) - (t_1 + 2d) = -20 - 5$$
$$5d = -25$$
$$d = -5$$

Step 3: Substitute $d = -5$ into equation (1).

$$t_1 + 2(-5) = 5$$
$$t_1 - 10 = 5$$
$$t_1 = 15$$

Step 4: Write the general term.

$$t_n = 15 + (n-1)(-5) = 15 - 5n + 5 = 20 - 5n$$

$$\boxed{t_1 = 15, \quad d = -5, \quad t_n = 20 - 5n}$$

7. Sequence: 5, 11, 17, 23, ... Is 200 a term?
Step 1: Identify t_1 and d.

$$t_1 = 5 \qquad d = 11 - 5 = 6$$

Step 2: Write the general term.

$$t_n = 5 + (n-1)(6) = 5 + 6n - 6 = 6n - 1$$

Step 3: Set $t_n = 200$ and solve for n.

$$6n - 1 = 200$$
$$6n = 201$$
$$n = \frac{201}{6} = 33.5$$

Step 4: Since $n = 33.5$ is not a whole number, 200 is not a term.

$$\boxed{200 \text{ is } \textbf{not} \text{ a term of this sequence.}}$$

8. Sequence: 4, 9, 14, 19, ... Is 84 a term?
Step 1: Identify t_1 and d.

$$t_1 = 4 \qquad d = 9 - 4 = 5$$

Step 2: Write the general term.

$$t_n = 4 + (n-1)(5) = 4 + 5n - 5 = 5n - 1$$

Step 3: Set $t_n = 84$ and solve for n.

$$5n - 1 = 84$$
$$5n = 85$$
$$n = 17$$

Step 4: Since $n = 17$ is a positive whole number, 84 is a term. Verify.

$$t_{17} = 5(17) - 1 = 85 - 1 = 84\checkmark$$

84 is the 17th term of the sequence.

9. (Challenge) Insert three arithmetic means between 8 and 28.
Step 1: Understand the structure.
Inserting three terms between 8 and 28 creates a sequence of five equally-spaced terms:

$$8,\ \_,\ \_,\ \_,\ 28$$

So $t_1 = 8$ and $t_5 = 28$.
Step 2: Use the general term formula to find d.

$$t_5 = t_1 + 4d$$
$$28 = 8 + 4d$$
$$20 = 4d$$
$$d = 5$$

Step 3: List all five terms.

$$t_1 = 8, \quad t_2 = 13, \quad t_3 = 18, \quad t_4 = 23, \quad t_5 = 28$$

The three arithmetic means are 13, 18, and 23.

10. (Challenge) First three terms: $x+2,\ 2x-1,\ 4x-3$.
Step 1: Use the property of arithmetic sequences.
In any arithmetic sequence, the common difference is constant. This means:

$$t_2 - t_1 = t_3 - t_2$$

Step 2: Substitute the expressions and set them equal.

$$(2x-1) - (x+2) = (4x-3) - (2x-1)$$

Step 3: Simplify both sides.
Left side:

$$2x - 1 - x - 2 = x - 3$$

Right side:

$$4x - 3 - 2x + 1 = 2x - 2$$

Step 4: Solve for x.

$$x - 3 = 2x - 2$$
$$-3 + 2 = 2x - x$$
$$-1 = x$$

Step 5: Find the first three terms by substituting $x = -1$.

$$t_1 = (-1) + 2 = 1$$
$$t_2 = 2(-1) - 1 = -3$$
$$t_3 = 4(-1) - 3 = -7$$

The common difference is $d = -3 - 1 = -4$.
Step 6: Find t_{10}.

$$t_{10} = 1 + (10-1)(-4) = 1 + 9 \times (-4) = 1 - 36$$

$$\boxed{x = -1, \quad t_1 = 1,\ t_2 = -3,\ t_3 = -7, \quad d = -4, \quad t_{10} = -35}$$

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