Arithmetic Series
ApexMath

Notes

What Is an Arithmetic Series?

A **series** is the sum of the terms of a sequence. An **arithmetic series** is the sum of the terms of an arithmetic sequence. We write the sum of the first n terms as S_n.

For example, the arithmetic sequence $3,\ 7,\ 11,\ 15,\ 19$ has a series:

$$S_5 = 3 + 7 + 11 + 15 + 19 = 55$$

For short sequences you can just add the terms. For long sequences — such as the sum of the first 100 terms — you need a formula.

The Two Sum Formulas

There are two equivalent formulas for the sum of the first n terms of an arithmetic series. You can use whichever one fits the information you have.

Formula 1 — Use this when you know t_1, d, and n:

$$\boxed{S_n = \frac{n}{2}\big(2t_1 + (n-1)\,d\big)}$$

Formula 2 — Use this when you know t_1, the last term t_n, and n:

$$\boxed{S_n = \frac{n}{2}\,(t_1 + t_n)}$$

Both formulas give the same result. Formula 2 is easier to use when the last term is already known. Formula 1 is more general and works whenever you know d.

Where Do These Formulas Come From?

Here is a short intuitive argument for Formula 2. Write the sum forwards and backwards:

$$S_n = t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + t_n$$
$$S_n = t_n + (t_n - d) + (t_n - 2d) + \cdots + t_1$$

Adding these two lines together, every pair of terms adds up to $(t_1 + t_n)$, and there are n such pairs:

$$2S_n = n\,(t_1 + t_n) \qquad \Rightarrow \qquad S_n = \frac{n}{2}\,(t_1 + t_n)$$

Solving for n When S_n Is Given

Sometimes you are told the value of S_n and asked to find n. Substitute the known values into one of the formulas, set it equal to the given sum, and solve the resulting quadratic equation. Only accept a positive whole number as your answer.

Finding t_n From a Formula for S_n

If you are given a formula for S_n (e.g. $S_n = 3n^2 + 2n$), you can find any individual term using the relationship:

$$t_n = S_n - S_{n-1} \qquad \text{for } n \geq 2$$

And the first term is simply $t_1 = S_1$.

Pro Tip

- **Choose the right formula for what you know.** If you know d but not t_n, use Formula 1. If you already know the last term t_n, Formula 2 is faster.
- **When solving for n, expand carefully and rearrange into standard quadratic form $an^2 + bn + c = 0$ before using the quadratic formula or factoring.** Trying to solve it without rearranging first leads to errors.
- **Always reject negative or fractional values of n.** The number of terms must be a positive whole number.
- **When using $t_n = S_n - S_{n-1}$, always verify that your formula also gives the correct t_1 by checking $t_1 = S_1$.**
- **A series and a sequence are different things.** A sequence lists terms separated by commas. A series adds them with + signs. S_n is always a single number — the running total.

Examples

Example 1 (Using Formula 1): Find the sum of the first 10 terms of the arithmetic sequence $3,\ 7,\ 11,\ 15,\ \ldots$

Step 1: Identify t_1, d, and n.

$$t_1 = 3 \qquad d = 7 - 3 = 4 \qquad n = 10$$

Step 2: Write Formula 1 and substitute.

$$S_n = \frac{n}{2}\big(2t_1 + (n-1)\,d\big)$$

$$S_{10} = \frac{10}{2}\big(2(3) + (10-1)(4)\big)$$

Step 3: Simplify inside the brackets first.

$$S_{10} = 5\big(6 + 36\big) = 5 \times 42$$

$$\boxed{S_{10} = 210}$$

Example 2 (Using Formula 2 — the famous sum): Find the sum $1+2+3+\cdots+100$.

Step 1: Identify t_1, t_n, and n.

$$t_1 = 1 \qquad t_{100} = 100 \qquad n = 100$$

Step 2: Use Formula 2 since we know the first and last terms.

$$S_n = \frac{n}{2}\left(t_1 + t_n\right)$$

$$S_{100} = \frac{100}{2}\left(1 + 100\right) = 50 \times 101$$

$$\boxed{S_{100} = 5050}$$

Example 3 (Solving for n): An arithmetic sequence has $t_1 = 3$ and $d = 5$. How many terms must be added to get a sum of 126?

Step 1: Write Formula 1 with the known values and set equal to 126.

$$S_n = \frac{n}{2}\big(2(3) + (n-1)(5)\big) = 126$$

Step 2: Simplify inside the brackets.

$$\frac{n}{2}\big(6 + 5n - 5\big) = 126$$

$$\frac{n}{2}(5n+1) = 126$$

Step 3: Multiply both sides by 2 to clear the fraction.

$$n(5n+1) = 252$$

$$5n^2 + n = 252$$

Step 4: Rearrange into standard quadratic form.

$$5n^2 + n - 252 = 0$$

Step 5: Solve by factoring. We need two numbers that multiply to $5 \times (-252) = -1260$ and add to 1. Those numbers are 36 and -35.

$$5n^2 + 36n - 35n - 252 = 0$$

$$n(5n+36) - 7(5n+36) = 0$$

$$(n-7)(5n+36) = 0$$

$$n = 7 \qquad \text{or} \qquad n = -\frac{36}{5}$$

Step 6: Reject $n = -\frac{36}{5}$ since n must be a positive whole number.

$$\text{Verify: } S_7 = \frac{7}{2}\big(2(3) + 6(5)\big) = \frac{7}{2}(36) = 7 \times 18 = 126\checkmark$$

$$\boxed{n = 7 \text{ terms}}$$

Example 4 (Negative first term): Find S_{20} for the sequence $-8,\ -3,\ 2,\ 7,\ \ldots$

Step 1: Identify t_1, d, and n.

$$t_1 = -8 \qquad d = -3 - (-8) = 5 \qquad n = 20$$

Step 2: Find t_{20} to use Formula 2 (since we know both ends).

$$t_{20} = -8 + (20 - 1)(5) = -8 + 95 = 87$$

Step 3: Apply Formula 2.

$$S_{20} = \frac{20}{2}\,(-8 + 87) = 10 \times 79$$

$$\boxed{S_{20} = 790}$$

Example 5 (Finding t_n from a formula for S_n): The sum of the first n terms of a sequence is given by $S_n = 3n^2 + 2n$. Find t_1, t_2, d, and the general term t_n.

Step 1: Find t_1 using $t_1 = S_1$.

$$t_1 = S_1 = 3(1)^2 + 2(1) = 3 + 2 = 5$$

Step 2: Find t_2 using $t_2 = S_2 - S_1$.

$$S_2 = 3(2)^2 + 2(2) = 12 + 4 = 16$$
$$t_2 = S_2 - S_1 = 16 - 5 = 11$$

Step 3: Find d.

$$d = t_2 - t_1 = 11 - 5 = 6$$

Step 4: Find the general term t_n for $n \geq 2$ using $t_n = S_n - S_{n-1}$.

$$S_{n-1} = 3(n-1)^2 + 2(n-1) = 3(n^2 - 2n + 1) + 2n - 2 = 3n^2 - 4n + 1$$

$$t_n = S_n - S_{n-1} = (3n^2 + 2n) - (3n^2 - 4n + 1) = 6n - 1$$

Check at $n = 1$: $t_1 = 6(1) - 1 = 5$ ✓

$$\boxed{t_1 = 5, \quad d = 6, \quad t_n = 6n - 1}$$

Common Mistakes

- **Confusing S_n and t_n.** S_n is a running total — the sum of all terms up to position n. t_n is just one term. They are very different things.
- **Forgetting to divide by 2 in the formula.** Both formulas have $\dfrac{n}{2}$ at the front. Leaving out the division by 2 doubles the answer.
- **Using n instead of $(n-1)$ inside the bracket in Formula 1.** The bracket contains $(n-1)\,d$, not $n \cdot d$.
- **Not rearranging to standard quadratic form before solving for n.** Trying to solve $\frac{n}{2}(2t_1+(n-1)d) = S_n$ directly without expanding leads to errors. Always expand fully first.
- **Accepting a negative or non-integer value of n.** The number of terms in a series must be a positive whole number. If your quadratic gives two solutions, one will always be negative or fractional — reject it.

Worksheet

1. Find S_{15} for the arithmetic sequence with $t_1 = 2$ and $d = 3$.

2. Find S_{20} for the arithmetic sequence with $t_1 = 100$ and $d = -4$.

3. Find the sum $5 + 9 + 13 + \cdots + 53$.
   (Find n first using the general term formula.)

4. The arithmetic sequence has $t_1 = 3$ and $d = 4$. How many terms must be added to reach a sum of 820?

5. The arithmetic sequence has $t_1 = 6$ and $d = 4$. How many terms must be added to reach a sum of 240?

6. Find S_{30} for the sequence $-5,\ -1,\ 3,\ 7,\ \ldots$

7. Find the sum of all integers from 1 to 200 inclusive.

8. Find the sum of all odd integers from 1 to 99 inclusive.
   (Find n first, then apply a sum formula.)

9. *(Challenge)* The sum of the first n terms of a sequence is given by $S_n = 2n^2 + 5n$. Find t_1, d, and the general term t_n.

10. *(Challenge)* A theatre has 18 rows of seats. The first row has 20 seats, and each row after that has 3 more seats than the row before it. Find the total number of seats in the theatre.

Answer Key

1. $t_1 = 2$, $d = 3$, $n = 15$
Step 1: Apply Formula 1.

$$S_n = \frac{n}{2}\big(2t_1 + (n-1)\,d\big)$$

Step 2: Substitute and simplify inside the bracket first.

$$S_{15} = \frac{15}{2}\big(2(2) + (14)(3)\big) = \frac{15}{2}\big(4 + 42\big) = \frac{15}{2} \times 46$$

$$S_{15} = 15 \times 23$$

$$\boxed{S_{15} = 345}$$

2. $t_1 = 100$, $d = -4$, $n = 20$
Step 1: Find t_{20} so we can use Formula 2.

$$t_{20} = 100 + (20-1)(-4) = 100 - 76 = 24$$

Step 2: Apply Formula 2.

$$S_{20} = \frac{20}{2}\,(t_1 + t_{20}) = 10\,(100 + 24) = 10 \times 124$$

$$\boxed{S_{20} = 1240}$$

3. Sum $5 + 9 + 13 + \cdots + 53$
Step 1: Identify t_1 and d.

$$t_1 = 5 \qquad d = 9 - 5 = 4$$

Step 2: Find n by setting $t_n = 53$.

$$5 + (n-1)(4) = 53$$
$$(n-1)(4) = 48$$
$$n - 1 = 12 \qquad \Rightarrow \qquad n = 13$$

Step 3: Apply Formula 2 (we know both ends).

$$S_{13} = \frac{13}{2}\,(5 + 53) = \frac{13}{2} \times 58 = 13 \times 29$$

$$\boxed{S_{13} = 377}$$

4. $t_1 = 3$, $d = 4$, $S_n = 820$

Step 1: Write Formula 1 and set equal to 820**.**

$$\frac{n}{2}\big(2(3) + (n-1)(4)\big) = 820$$

Step 2: Simplify inside the bracket.

$$\frac{n}{2}(6 + 4n - 4) = 820$$

$$\frac{n}{2}(4n + 2) = 820$$

Step 3: Multiply both sides by 2.

$$n(4n + 2) = 1640$$

$$4n^2 + 2n = 1640$$

Step 4: Divide every term by 2, then rearrange.

$$2n^2 + n = 820$$

$$2n^2 + n - 820 = 0$$

Step 5: Solve by factoring. We need factors of $2 \times (-820) = -1640$ **that add to** 1**. Those are** 41 **and** -40**.**

$$2n^2 + 41n - 40n - 820 = 0$$

$$n(2n + 41) - 20(2n + 41) = 0$$

$$(n - 20)(2n + 41) = 0$$

$$n = 20 \qquad \text{or} \qquad n = -\frac{41}{2}$$

Step 6: Reject the negative value. Verify.

$$S_{20} = \frac{20}{2}\big(2(3) + 19(4)\big) = 10(6 + 76) = 10 \times 82 = 820\checkmark$$

$$\boxed{n = 20 \text{ terms}}$$

5. $t_1 = 6$, $d = 4$, $S_n = 240$

Step 1: Write Formula 1 and set equal to 240**.**

$$\frac{n}{2}\big(2(6) + (n-1)(4)\big) = 240$$

Step 2: Simplify inside the bracket.

$$\frac{n}{2}(12 + 4n - 4) = 240$$

$$\frac{n}{2}(4n + 8) = 240$$

Step 3: Multiply both sides by 2.

$$n(4n + 8) = 480$$

$$4n^2 + 8n = 480$$

Step 4: Divide every term by 4, then rearrange.

$$n^2 + 2n = 120$$

$$n^2 + 2n - 120 = 0$$

Step 5: Factor.

$$(n + 12)(n - 10) = 0$$

$$n = 10 \qquad \text{or} \qquad n = -12$$

Step 6: Reject $n = -12$**. Verify.**

$$S_{10} = \frac{10}{2}\big(2(6) + 9(4)\big) = 5(12 + 36) = 5 \times 48 = 240\checkmark$$

$$\boxed{n = 10 \text{ terms}}$$

6. Sequence: $-5,\ -1,\ 3,\ 7,\ \ldots$ Find S_{30}.
Step 1: Identify t_1, d, and n.

$$t_1 = -5 \qquad d = -1 - (-5) = 4 \qquad n = 30$$

Step 2: Find t_{30}.

$$t_{30} = -5 + (30-1)(4) = -5 + 116 = 111$$

Step 3: Apply Formula 2.

$$S_{30} = \frac{30}{2}(-5 + 111) = 15 \times 106$$

$$\boxed{S_{30} = 1590}$$

7. Sum of all integers from 1 to 200.
Step 1: Identify $t_1 = 1$, $t_{200} = 200$, $n = 200$.
Step 2: Apply Formula 2.

$$S_{200} = \frac{200}{2}(1 + 200) = 100 \times 201$$

$$\boxed{S_{200} = 20100}$$

8. Sum $1 + 3 + 5 + \cdots + 99$
Step 1: Identify $t_1 = 1$, $d = 2$.
Step 2: Find n by setting $t_n = 99$.

$$1 + (n-1)(2) = 99$$

$$(n-1)(2) = 98$$

$$n - 1 = 49 \qquad \Rightarrow \qquad n = 50$$

Step 3: Apply Formula 2.

$$S_{50} = \frac{50}{2}(1 + 99) = 25 \times 100$$

$$\boxed{S_{50} = 2500}$$

9. (Challenge) $S_n = 2n^2 + 5n$
Step 1: Find t_1 using $t_1 = S_1$.

$$t_1 = 2(1)^2 + 5(1) = 2 + 5 = 7$$

Step 2: Find t_2 using $t_2 = S_2 - S_1$.

$$S_2 = 2(2)^2 + 5(2) = 8 + 10 = 18$$
$$t_2 = 18 - 7 = 11$$

Step 3: Find d.

$$d = t_2 - t_1 = 11 - 7 = 4$$

Step 4: Find the general term $t_n = S_n - S_{n-1}$ for $n \geq 2$.
First expand S_{n-1}:

$$S_{n-1} = 2(n-1)^2 + 5(n-1) = 2(n^2 - 2n + 1) + 5n - 5 = 2n^2 + n - 3$$

Then subtract:

$$t_n = S_n - S_{n-1} = (2n^2 + 5n) - (2n^2 + n - 3) = 4n + 3$$

Step 5: Verify at $n = 1$: $t_1 = 4(1) + 3 = 7$ ✓

$$\boxed{t_1 = 7, \quad d = 4, \quad t_n = 4n + 3}$$

10. (Challenge) Theatre with 18 rows, $t_1 = 20$, $d = 3$.
Step 1: Identify the known values.

$$t_1 = 20 \qquad d = 3 \qquad n = 18$$

Step 2: Find the number of seats in the last row, t_{18}.

$$t_{18} = 20 + (18 - 1)(3) = 20 + 51 = 71$$

Step 3: Apply Formula 2 to find the total number of seats.

$$S_{18} = \frac{18}{2}(20 + 71) = 9 \times 91$$

$$\boxed{S_{18} = 819 \text{ seats in total}}$$

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