Geometric Sequences
ApexMath

Notes

What Is a Geometric Sequence?

A **geometric sequence** is a list of numbers where each term is found by *multiplying* the previous term by the same fixed number. That fixed number is called the **common ratio** and is written as r.

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \cdots$$

To find r, divide any term by the term directly before it.

Examples of Geometric Sequences

- $3,\ 6,\ 12,\ 24,\ 48,\ \ldots$ Here $r = 2$ (multiply by 2 each time)
- $81,\ 27,\ 9,\ 3,\ 1,\ \ldots$ Here $r = \dfrac{1}{3}$ (multiply by $\frac{1}{3}$ each time)
- $2,\ -6,\ 18,\ -54,\ \ldots$ Here $r = -3$ (the terms alternate in sign)

The General Term Formula

The nth term of a geometric sequence is:

$$\boxed{t_n = t_1 \cdot r^{n-1}}$$

Where:

- t_n is the term you want to find
- t_1 is the first term of the sequence
- r is the common ratio
- n is the position number of the term

Why r^{n-1} and not r^n?

To travel from t_1 to t_2 you multiply by r once. To travel from t_1 to t_3 you multiply by r twice. To travel from t_1 to t_n you multiply by r exactly $(n-1)$ times. You are already at t_1, so you do not multiply at all to stay there — that is why $t_1 = t_1 \cdot r^0 = t_1 \cdot 1 = t_1$.

Negative Common Ratios

When r is negative, the terms alternate between positive and negative. For example, with $t_1 = 2$ and $r = -3$:

$$2,\ -6,\ 18,\ -54,\ 162,\ \ldots$$

The formula still works exactly the same way. Just be careful with signs when you raise a negative number to a power: a negative number raised to an *even* power is positive, and raised to an *odd* power is negative.

Finding t_1 and r When Given Two Terms

If you are given two non-consecutive terms, you can find t_1 and r by writing two equations and dividing them. For example, if $t_3 = 12$ and $t_6 = 96$:

$$t_1 \cdot r^2 = 12 \qquad \text{and} \qquad t_1 \cdot r^5 = 96$$

Dividing the second equation by the first eliminates t_1 and leaves only a power of r, which you can then solve. This method is shown in full in Example 5.

Is a Given Value a Term of the Sequence?

Set t_n equal to the target value and solve for n. If n is a positive whole number, the value is a term. If n is not a whole number, it is not a term. For geometric sequences, solving for n may require thinking about powers: ask yourself what power of r gives the ratio between the target value and t_1.

Pro Tip

- **Always find r by dividing, not subtracting.** Dividing consecutive terms gives the common ratio. Subtracting them gives nothing useful for geometric sequences.
- **The exponent is $n-1$, not n.** Writing $t_n = t_1 \cdot r^n$ is the single most common error. Always use r^{n-1}.
- **When given two non-consecutive terms, divide the larger-indexed equation by the smaller-indexed one.** This cancels t_1 cleanly and leaves a power of r that you can solve.
- **Watch signs carefully with negative ratios.** Even powers of a negative number are positive; odd powers are negative. Write each step out fully rather than trying to do it mentally.
- **r can be a fraction.** A decreasing geometric sequence has $0 < r < 1$. The formula works identically — just be careful with fraction arithmetic when evaluating powers.

Examples

Example 1 (Listing terms from t_1 and r): Write the first five terms of the geometric sequence with $t_1 = 3$ and $r = 2$.

Step 1: Start with t_1.

$$t_1 = 3$$

Step 2: Multiply by $r = 2$ each time.

$$t_2 = 3 \times 2 = 6$$
$$t_3 = 6 \times 2 = 12$$
$$t_4 = 12 \times 2 = 24$$
$$t_5 = 24 \times 2 = 48$$

$$\boxed{3,\ 6,\ 12,\ 24,\ 48}$$

Example 2 (Decreasing sequence): Write the first five terms of the geometric sequence with $t_1 = 96$ and $r = \frac{1}{2}$.

Step 1: Start with t_1.

$$t_1 = 96$$

Step 2: Multiply by $\frac{1}{2}$ each time (i.e. divide by 2).

$$t_2 = 96 \times \frac{1}{2} = 48$$
$$t_3 = 48 \times \frac{1}{2} = 24$$
$$t_4 = 24 \times \frac{1}{2} = 12$$
$$t_5 = 12 \times \frac{1}{2} = 6$$

$$\boxed{96,\ 48,\ 24,\ 12,\ 6}$$

Example 3 (Finding the general term and a specific term): The sequence is $5,\ 15,\ 45,\ 135,\ \ldots$ Find the general term t_n and use it to find t_8.

Step 1: Identify t_1 and r.

$$t_1 = 5 \qquad r = \frac{15}{5} = 3$$

Step 2: Write the general term formula and substitute.

$$t_n = t_1 \cdot r^{\,n-1} = 5 \cdot 3^{\,n-1}$$

Step 3: Find t_8 by substituting $n = 8$.

$$t_8 = 5 \cdot 3^{\,8-1} = 5 \cdot 3^7$$

Step 4: Evaluate 3^7, then multiply.

$$3^7 = 2187 \qquad \Rightarrow \qquad t_8 = 5 \times 2187$$

$$\boxed{t_n = 5 \cdot 3^{\,n-1}, \qquad t_8 = 10935}$$

Example 4 (Negative common ratio): Write the first five terms of the geometric sequence with $t_1 = 2$ and $r = -3$.

Step 1: Start with t_1.

$$t_1 = 2$$

Step 2: Multiply by $r = -3$ each time. Watch the signs carefully.

$$t_2 = 2 \times (-3) = -6$$
$$t_3 = -6 \times (-3) = 18$$
$$t_4 = 18 \times (-3) = -54$$
$$t_5 = -54 \times (-3) = 162$$

Notice that the terms alternate between positive and negative because r is negative.

$$\boxed{2,\ -6,\ 18,\ -54,\ 162}$$

Example 5 (Finding t_1 and r from two terms): A geometric sequence has $t_3 = 12$ and $t_6 = 96$. Find t_1, r, and t_{10}.

Step 1: Write the formula for each given term.
For t_3:

$$t_1 \cdot r^2 = 12 \qquad \cdots \ (1)$$

For t_6:

$$t_1 \cdot r^5 = 96 \qquad \cdots \ (2)$$

Step 2: Divide equation (2) by equation (1) to eliminate t_1.

$$\frac{t_1 \cdot r^5}{t_1 \cdot r^2} = \frac{96}{12}$$
$$r^3 = 8$$
$$r = 2$$

Step 3: Substitute $r = 2$ into equation (1) to find t_1.

$$t_1 \cdot (2)^2 = 12$$
$$t_1 \cdot 4 = 12$$
$$t_1 = 3$$

Step 4: Find t_{10}.

$$t_{10} = 3 \cdot 2^{10-1} = 3 \cdot 2^9 = 3 \times 512$$

$$\boxed{t_1 = 3, \quad r = 2, \quad t_{10} = 1536}$$

Example 6 (Is a value a term of the sequence?): The sequence is $3,\ 6,\ 12,\ 24,\ \ldots$ Is 384 a term?

Step 1: Identify t_1 and r.

$$t_1 = 3 \qquad r = \frac{6}{3} = 2$$

Step 2: Write the general term.

$$t_n = 3 \cdot 2^{n-1}$$

Step 3: Set $t_n = 384$ and solve.

$$3 \cdot 2^{n-1} = 384$$
$$2^{n-1} = \frac{384}{3} = 128$$

Step 4: Express 128 as a power of 2.

$$128 = 2^7 \qquad \Rightarrow \qquad n - 1 = 7 \qquad \Rightarrow \qquad n = 8$$

Step 5: Since $n = 8$ is a positive whole number, 384 is a term. Verify.

$$t_8 = 3 \cdot 2^7 = 3 \times 128 = 384\checkmark$$

$$\boxed{\text{384 is the 8th term of the sequence.}}$$

Common Mistakes

- **Writing $t_n = t_1 \cdot r^n$ instead of $t_n = t_1 \cdot r^{n-1}$.** Using r^n shifts every term one position too far. Always use r^{n-1}.
- **Finding r by subtracting instead of dividing.** For geometric sequences, the ratio is $r = t_2 \div t_1$, not $t_2 - t_1$.
- **Dropping the negative sign on r.** If the sequence alternates in sign, r is negative. Forgetting the negative gives a sequence that never changes sign.
- **Adding instead of multiplying when listing terms.** Each new term is found by *multiplying* the previous term by r, not adding r.
- **When solving for n, not checking whether n is a whole number.** If n is not a positive integer, the value is not a term of the sequence.
- **When given two terms, subtracting the equations instead of dividing.** Subtracting does not eliminate t_1 cleanly in a geometric sequence. Always divide the equations.

Worksheet

1. The sequence $4,\ 12,\ 36,\ 108,\ \ldots$ is geometric.
   (a) State t_1 and r.
   (b) Write the general term t_n.
   (c) Find t_7.

2. The sequence $160,\ 80,\ 40,\ 20,\ \ldots$ is geometric.
   (a) State t_1 and r.
   (b) Write the general term t_n.
   (c) Find t_8.

3. A geometric sequence has $t_1 = 5$ and $r = 4$.
   (a) Write the general term t_n.
   (b) Find t_6.

4. A geometric sequence has $t_1 = 729$ and $r = \dfrac{1}{3}$.
   (a) Write the general term t_n.
   (b) Find t_7.

5. A geometric sequence has $t_1 = 6$ and $r = -2$.
   (a) List the first five terms.
   (b) Find t_8.

6. A geometric sequence has $t_2 = 18$ and $t_5 = 486$. Find t_1, r, and t_8.

7. A geometric sequence has $t_3 = 45$ and $t_6 = 1215$. Find t_1, r, and the general term t_n.

8. The sequence $2,\ 6,\ 18,\ 54,\ \ldots$ is geometric. Is 1458 a term of this sequence? If so, which term is it?

9. The sequence $4,\ 20,\ 100,\ 500,\ \ldots$ is geometric. Is 500 a term of this sequence? If so, which term is it?

10. *(Challenge)* Insert three geometric means between 2 and 162.
(Hint: You need five equally-spaced terms in total, with 2 *as the first and* 162 *as the last. Find r first.)*

Answer Key

1. Sequence: $4,\ 12,\ 36,\ 108,\ \ldots$

(a) State t_1 and r.

$$t_1 = 4 \qquad r = \frac{12}{4} = 3$$

$$\boxed{t_1 = 4, \quad r = 3}$$

(b) General term.
Step 1: Substitute into the formula.

$$t_n = t_1 \cdot r^{\,n-1} = 4 \cdot 3^{\,n-1}$$

$$\boxed{t_n = 4 \cdot 3^{\,n-1}}$$

(c) Find t_7.
Step 1: Substitute $n = 7$.

$$t_7 = 4 \cdot 3^{\,7-1} = 4 \cdot 3^6$$

Step 2: Evaluate $3^6 = 729$, then multiply.

$$t_7 = 4 \times 729$$

$$\boxed{t_7 = 2916}$$

2. Sequence: $160,\ 80,\ 40,\ 20,\ \ldots$

(a) State t_1 and r.

$$t_1 = 160 \qquad r = \frac{80}{160} = \frac{1}{2}$$

$$\boxed{t_1 = 160, \quad r = \frac{1}{2}}$$

(b) General term.

$$t_n = 160 \cdot \left(\frac{1}{2}\right)^{n-1}$$

$$\boxed{t_n = 160 \cdot \left(\frac{1}{2}\right)^{n-1}}$$

(c) Find t_8.
Step 1: Substitute $n = 8$.

$$t_8 = 160 \cdot \left(\frac{1}{2}\right)^{7} = \frac{160}{2^7} = \frac{160}{128}$$

Step 2: Simplify the fraction.

$$t_8 = \frac{160}{128} = \frac{5}{4}$$

$$\boxed{t_8 = \frac{5}{4}}$$

3. $t_1 = 5$, $r = 4$
(a) General term.

$$t_n = 5 \cdot 4^{n-1}$$

$$\boxed{t_n = 5 \cdot 4^{n-1}}$$

(b) Find t_6.
Step 1: Substitute $n = 6$.

$$t_6 = 5 \cdot 4^{5}$$

Step 2: Evaluate $4^5 = 1024$, then multiply.

$$t_6 = 5 \times 1024$$

$$\boxed{t_6 = 5120}$$

4. $t_1 = 729$, $r = \dfrac{1}{3}$
(a) General term.

$$t_n = 729 \cdot \left(\frac{1}{3}\right)^{n-1}$$

$$\boxed{t_n = 729 \cdot \left(\frac{1}{3}\right)^{n-1}}$$

(b) Find t_7.
Step 1: Substitute $n = 7$.

$$t_7 = 729 \cdot \left(\frac{1}{3}\right)^{6} = \frac{729}{3^6}$$

Step 2: Evaluate $3^6 = 729$.

$$t_7 = \frac{729}{729}$$

$$\boxed{t_7 = 1}$$

5. $t_1 = 6$, $r = -2$
(a) First five terms.
Step 1: Start with $t_1 = 6$ and multiply by -2 each time.

$$\begin{aligned} t_1 &= 6 \\ t_2 &= 6 \times (-2) = -12 \\ t_3 &= -12 \times (-2) = 24 \\ t_4 &= 24 \times (-2) = -48 \\ t_5 &= -48 \times (-2) = 96 \end{aligned}$$

$$\boxed{6,\ -12,\ 24,\ -48,\ 96}$$

(b) Find t_8.
Step 1: Use the formula $t_n = t_1 \cdot r^{\,n-1}$.

$$t_8 = 6 \cdot (-2)^{\,7}$$

Step 2: Evaluate $(-2)^7$. The exponent 7 is odd, so the result is negative.

$$(-2)^7 = -128$$

$$t_8 = 6 \times (-128)$$

$$\boxed{t_8 = -768}$$

6. $t_2 = 18$, $t_5 = 486$

Step 1: Write the formula for each given term.

For t_2:

$$t_1 \cdot r = 18 \qquad \cdots \ (1)$$

For t_5:

$$t_1 \cdot r^4 = 486 \qquad \cdots \ (2)$$

Step 2: Divide equation (2) by equation (1) to eliminate t_1.

$$\frac{t_1 \cdot r^4}{t_1 \cdot r} = \frac{486}{18}$$

$$r^3 = 27$$

$$r = 3$$

Step 3: Substitute $r = 3$ into equation (1) to find t_1.

$$t_1 \cdot 3 = 18$$

$$t_1 = 6$$

Step 4: Find t_8.

$$t_8 = 6 \cdot 3^7 = 6 \times 2187$$

$$\boxed{t_1 = 6, \quad r = 3, \quad t_8 = 13122}$$

7. $t_3 = 45$, $t_6 = 1215$

Step 1: Write the formula for each given term.

For t_3:

$$t_1 \cdot r^2 = 45 \qquad \cdots \ (1)$$

For t_6:

$$t_1 \cdot r^5 = 1215 \qquad \cdots \ (2)$$

Step 2: Divide equation (2) by equation (1).

$$\frac{t_1 \cdot r^5}{t_1 \cdot r^2} = \frac{1215}{45}$$

$$r^3 = 27$$

$$r = 3$$

Step 3: Substitute $r = 3$ into equation (1).

$$t_1 \cdot (3)^2 = 45$$

$$t_1 \cdot 9 = 45$$

$$t_1 = 5$$

Step 4: Write the general term.

$$t_n = 5 \cdot 3^{n-1}$$

$$\boxed{t_1 = 5, \quad r = 3, \quad t_n = 5 \cdot 3^{n-1}}$$

8. Sequence: $2, 6, 18, 54, \ldots$ Is 1458 a term?
Step 1: Identify t_1 and r.

$$t_1 = 2 \qquad r = \frac{6}{2} = 3$$

Step 2: Write the general term.

$$t_n = 2 \cdot 3^{n-1}$$

Step 3: Set $t_n = 1458$ and solve.

$$2 \cdot 3^{n-1} = 1458$$
$$3^{n-1} = \frac{1458}{2} = 729$$

Step 4: Express 729 as a power of 3.

$$729 = 3^6 \qquad \Rightarrow \qquad n - 1 = 6 \qquad \Rightarrow \qquad n = 7$$

Step 5: Since $n = 7$ is a positive whole number, verify.

$$t_7 = 2 \cdot 3^6 = 2 \times 729 = 1458 \checkmark$$

$$\boxed{\text{1458 is the 7th term of the sequence.}}$$

9. Sequence: $4, 20, 100, 500, \ldots$ Is 500 a term?
Step 1: Identify t_1 and r.

$$t_1 = 4 \qquad r = \frac{20}{4} = 5$$

Step 2: Write the general term.

$$t_n = 4 \cdot 5^{n-1}$$

Step 3: Set $t_n = 500$ and solve.

$$4 \cdot 5^{n-1} = 500$$
$$5^{n-1} = \frac{500}{4} = 125$$

Step 4: Express 125 as a power of 5.

$$125 = 5^3 \qquad \Rightarrow \qquad n - 1 = 3 \qquad \Rightarrow \qquad n = 4$$

Step 5: Verify.

$$t_4 = 4 \cdot 5^3 = 4 \times 125 = 500 \checkmark$$

$$\boxed{\text{500 is the 4th term of the sequence.}}$$

10. (Challenge) Insert three geometric means between 2 and 162.

Step 1: Understand the structure.

Inserting three terms between 2 and 162 creates a sequence of five equally-spaced (in ratio) terms:

$$2,\ \_,\ \_,\ \_,\ 162$$

So $t_1 = 2$ and $t_5 = 162$.

Step 2: Use the general term formula to find r.

$$t_5 = t_1 \cdot r^4$$
$$162 = 2 \cdot r^4$$
$$r^4 = \frac{162}{2} = 81$$
$$r = \sqrt[4]{81} = 3$$

(Since we need positive terms, we take $r = 3$.)

Step 3: List all five terms.

$$t_1 = 2, \quad t_2 = 6, \quad t_3 = 18, \quad t_4 = 54, \quad t_5 = 162$$

$$\boxed{\text{The three geometric means are } 6,\ 18,\ \text{and } 54.}$$

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