Geometric Series
ApexMath

Notes

What Is a Geometric Series?

A **geometric series** is the sum of the terms of a geometric sequence. We write the sum of the first n terms as S_n.

For example, the geometric sequence $3,\ 6,\ 12,\ 24$ has a series:

$$S_4 = 3 + 6 + 12 + 24 = 45$$

For short sequences you can add directly. For longer sequences, a formula is needed.

The Sum Formula

The sum of the first n terms of a geometric series with first term t_1 and common ratio $r \neq 1$ is:

$$\boxed{S_n = \frac{t_1\left(r^n - 1\right)}{r - 1}}$$

This formula can also be written as:

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

Both forms are equivalent. The first form is easier when $r > 1$. The second form avoids a negative denominator when $r < 1$. You may use either form in your work.

Why Is There No Formula When $r = 1$?

If $r = 1$, every term equals t_1, and the sum is simply $S_n = n \cdot t_1$. The formula above breaks down when $r = 1$ because the denominator $(r - 1)$ would be zero. This case is trivial and rarely appears in problems.

Where Does the Formula Come From?

Write S_n and then write $r \cdot S_n$ directly below it:

$$\begin{aligned} S_n &= t_1 + t_1 r + t_1 r^2 + \cdots + t_1 r^{n-1} \\ rS_n &= \qquad\ t_1 r + t_1 r^2 + \cdots + t_1 r^{n-1} + t_1 r^n \end{aligned}$$

Subtract the first line from the second. Almost every term cancels:

$$rS_n - S_n = t_1 r^n - t_1$$

$$S_n(r - 1) = t_1(r^n - 1)$$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

Finding n When S_n Is Given

Sometimes you know the sum and need to find the number of terms. Substitute the known values into the formula, simplify, and isolate the power of r. Then express both sides as the same base and equate exponents. This only works cleanly when the answer is a whole-number power — always verify your result.

Pro Tip

- **Choose the form of the formula that avoids negatives.** When $r > 1$, use $\dfrac{t_1(r^n - 1)}{r-1}$. When $0 < r < 1$, use $\dfrac{t_1(1 - r^n)}{1-r}$. Either form works, but choosing wisely reduces sign errors.
- **Do not confuse S_n with t_n.** S_n is the running total of all terms up to position n. t_n is just one single term.
- **When solving for n, isolate the power of r completely before trying to identify the exponent.** Students who try to read off n before simplifying almost always make an error.
- **Always verify your answer by substituting back.** If $S_n = \frac{t_1(r^n-1)}{r-1}$ and you found $n = 7$, check that plugging in $n = 7$ gives the stated sum.
- **Watch signs carefully when r is negative.** Even powers of a negative r are positive; odd powers are negative. Expand carefully rather than doing this step mentally.

Examples

Example 1 (Direct application of the formula): Find S_6 for the geometric sequence with $t_1 = 3$ and $r = 2$.

Step 1: Identify t_1, r, and n.

$$t_1 = 3 \qquad r = 2 \qquad n = 6$$

Step 2: Substitute into the formula.

$$S_n = \frac{t_1\left(r^n - 1\right)}{r-1}$$

$$S_6 = \frac{3\left(2^6 - 1\right)}{2-1}$$

Step 3: Evaluate $2^6 = 64$, then simplify.

$$S_6 = \frac{3\left(64 - 1\right)}{1} = 3 \times 63$$

$$\boxed{S_6 = 189}$$

Example 2 (Fractional common ratio): Find S_5 for the geometric sequence with $t_1 = 80$ and $r = \dfrac{1}{2}$.

Step 1: Identify t_1, r, and n.

$$t_1 = 80 \qquad r = \frac{1}{2} \qquad n = 5$$

Step 2: Use the alternate form since $r < 1$.

$$S_5 = \frac{80\left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}}$$

Step 3: Evaluate $\left(\frac{1}{2}\right)^5 = \dfrac{1}{32}$.

$$S_5 = \frac{80\left(1 - \dfrac{1}{32}\right)}{\dfrac{1}{2}} = \frac{80 \times \dfrac{31}{32}}{\dfrac{1}{2}}$$

Step 4: Dividing by $\dfrac{1}{2}$ is the same as multiplying by 2.

$$S_5 = 80 \times \frac{31}{32} \times 2 = \frac{80 \times 31 \times 2}{32} = \frac{4960}{32}$$

$$\boxed{S_5 = 155}$$

(Quick check: $80 + 40 + 20 + 10 + 5 = 155$ ✓)

Example 3 (Find n from the last term, then sum): Find the sum $2 + 6 + 18 + \cdots + 1458$.

Step 1: Identify t_1 and r.

$$t_1 = 2 \qquad r = \frac{6}{2} = 3$$

Step 2: Find n by setting the general term equal to 1458.

$$t_n = 2 \cdot 3^{n-1} = 1458 \qquad \Rightarrow \qquad 3^{n-1} = \frac{1458}{2} = 729 = 3^6$$
$$n - 1 = 6 \qquad \Rightarrow \qquad n = 7$$

Step 3: Apply the formula with $n = 7$.

$$S_7 = \frac{2\,(3^7 - 1)}{3 - 1} = \frac{2\,(2187 - 1)}{2} = \frac{2 \times 2186}{2} = 2186$$

$$\boxed{S_7 = 2186}$$

Example 4 (Solving for n): A geometric series has $t_1 = 1$ and $r = 2$. How many terms are needed for $S_n = 255$?

Step 1: Substitute known values into the formula.

$$S_n = \frac{1\,(2^n - 1)}{2 - 1} = 2^n - 1 = 255$$

Step 2: Isolate the power of r.

$$2^n = 256$$

Step 3: Express 256 as a power of 2.

$$256 = 2^8 \qquad \Rightarrow \qquad n = 8$$

Step 4: Verify.

$$S_8 = \frac{1\,(2^8 - 1)}{1} = 256 - 1 = 255\checkmark$$

$$\boxed{n = 8 \text{ terms}}$$

Example 5 (Writing a general formula for S_n): A geometric sequence has $t_1 = 5$ and $r = 3$. Write a formula for S_n and use it to find S_4.

Step 1: Substitute $t_1 = 5$ and $r = 3$ into the formula.

$$S_n = \frac{5\,(3^n - 1)}{3 - 1} = \frac{5\,(3^n - 1)}{2}$$

Step 2: Find S_4 by substituting $n = 4$.

$$S_4 = \frac{5\,(3^4 - 1)}{2} = \frac{5\,(81 - 1)}{2} = \frac{5 \times 80}{2} = \frac{400}{2}$$

$$\boxed{S_n = \frac{5\,(3^n - 1)}{2}, \qquad S_4 = 200}$$

(Quick check: $5 + 15 + 45 + 135 = 200$ ✓)

Common Mistakes

- **Using r^n instead of $r^n - 1$ in the numerator.** The numerator is $(r^n - 1)$, not just r^n. Forgetting to subtract 1 gives an answer that is too large by t_1.
- **Using r instead of $(r - 1)$ in the denominator.** The denominator is $(r - 1)$. Writing just r in the denominator is one of the most common formula errors.
- **Forgetting to find n before summing.** When a series is given as $t_1 + \cdots + t_k$, you must find n using the general term formula before you can apply the sum formula.

- **Sign errors with negative r.** When r is negative, be very careful evaluating r^n. Write each step out fully.
- **Not verifying the answer when solving for n.** Because you are matching exponents, a small arithmetic mistake can give a plausible-looking but wrong value of n. Always substitute back to confirm.

Worksheet

1. Find S_7 for the geometric sequence with $t_1 = 1$ and $r = 3$.

2. Find S_8 for the geometric sequence with $t_1 = 5$ and $r = 2$.

3. Find the sum $4 + 12 + 36 + \cdots + 2916$.
   (Find n first using the general term formula.)

4. Find S_6 for the geometric sequence with $t_1 = 96$ and $r = \dfrac{1}{2}$.

5. Find S_5 for the geometric sequence with $t_1 = 2$ and $r = -3$.

6. A geometric series has $t_1 = 3$ and $r = 2$. How many terms are needed for $S_n = 381$?

7. A geometric series has $t_1 = 4$ and $r = 3$. How many terms are needed for $S_n = 4372$?

8. Find the sum $1 + 2 + 4 + \cdots + 512$.
   (Find n first, then apply the formula.)

9. Find S_5 for the geometric sequence with $t_1 = 192$ and $r = \frac{1}{2}$.

10. *(Challenge)* A ball is dropped from a height of 32 m. After each bounce it rises to $\frac{1}{2}$ of the height of the previous bounce. The ball bounces exactly 4 times before coming to rest. Find the total distance travelled by the ball.
(Hint: The ball falls 32 *m first. Then for each bounce it travels up* and *back down. The heights of the* 4 *bounces form a geometric sequence.)*

Answer Key

1. $t_1 = 1$, $r = 3$, $n = 7$
Step 1: Substitute into the formula.

$$S_7 = \frac{1\left(3^7 - 1\right)}{3 - 1} = \frac{3^7 - 1}{2}$$

Step 2: Evaluate $3^7 = 2187$.

$$S_7 = \frac{2187 - 1}{2} = \frac{2186}{2}$$

$$\boxed{S_7 = 1093}$$

2. $t_1 = 5$, $r = 2$, $n = 8$
Step 1: Substitute into the formula.

$$S_8 = \frac{5\left(2^8 - 1\right)}{2 - 1} = 5\left(2^8 - 1\right)$$

Step 2: Evaluate $2^8 = 256$.

$$S_8 = 5\left(256 - 1\right) = 5 \times 255$$

$$\boxed{S_8 = 1275}$$

3. Sum $4 + 12 + 36 + \cdots + 2916$
Step 1: Identify t_1 and r.

$$t_1 = 4 \qquad r = \frac{12}{4} = 3$$

Step 2: Find n by setting $t_n = 2916$.

$$4 \cdot 3^{\,n-1} = 2916 \qquad \Rightarrow \qquad 3^{\,n-1} = \frac{2916}{4} = 729 = 3^6$$
$$n - 1 = 6 \qquad \Rightarrow \qquad n = 7$$

Step 3: Apply the formula.

$$S_7 = \frac{4\left(3^7 - 1\right)}{3 - 1} = \frac{4\left(2187 - 1\right)}{2} = \frac{4 \times 2186}{2} = 2 \times 2186$$

$$\boxed{S_7 = 4372}$$

4. $t_1 = 96$, $r = \dfrac{1}{2}$, $n = 6$
Step 1: Use the alternate form since $r < 1$.

$$S_6 = \frac{96\left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}}$$

Step 2: Evaluate $\left(\frac{1}{2}\right)^6 = \dfrac{1}{64}$.

$$S_6 = \frac{96\left(1 - \dfrac{1}{64}\right)}{\dfrac{1}{2}} = \frac{96 \times \dfrac{63}{64}}{\dfrac{1}{2}}$$

Step 3: Simplify. Dividing by $\frac{1}{2}$ multiplies by 2.

$$S_6 = 96 \times \frac{63}{64} \times 2 = \frac{96 \times 63 \times 2}{64} = \frac{96}{64} \times 63 \times 2 = \frac{3}{2} \times 63 \times 2 = 3 \times 63$$

$$\boxed{S_6 = 189}$$

5. $t_1 = 2$, $r = -3$, $n = 5$
Step 1: Substitute into the formula. Be careful with the negative ratio.

$$S_5 = \frac{2\left((-3)^5 - 1\right)}{-3 - 1}$$

Step 2: Evaluate $(-3)^5$. The exponent is odd, so the result is negative.

$$(-3)^5 = -243$$

Step 3: Substitute and simplify.

$$S_5 = \frac{2\left(-243 - 1\right)}{-4} = \frac{2 \times (-244)}{-4} = \frac{-488}{-4}$$

$$\boxed{S_5 = 122}$$

6. $t_1 = 3$, $r = 2$, $S_n = 381$. Find n.
Step 1: Substitute into the formula and set equal to 381**.**

$$\frac{3\left(2^n - 1\right)}{2-1} = 381$$
$$3\left(2^n - 1\right) = 381$$

Step 2: Isolate the power.

$$2^n - 1 = \frac{381}{3} = 127$$
$$2^n = 128$$

Step 3: Express 128 **as a power of** 2**.**

$$128 = 2^7 \qquad \Rightarrow \qquad n = 7$$

Step 4: Verify.

$$S_7 = \frac{3\left(2^7 - 1\right)}{1} = 3 \times 127 = 381\checkmark$$

$$\boxed{n = 7 \text{ terms}}$$

7. $t_1 = 4$, $r = 3$, $S_n = 4372$. Find n.
Step 1: Substitute and set equal to 4372**.**

$$\frac{4\left(3^n - 1\right)}{3-1} = 4372$$
$$\frac{4\left(3^n - 1\right)}{2} = 4372$$
$$2\left(3^n - 1\right) = 4372$$

Step 2: Isolate the power.

$$3^n - 1 = \frac{4372}{2} = 2186$$
$$3^n = 2187$$

Step 3: Express 2187 **as a power of** 3**.**

$$2187 = 3^7 \qquad \Rightarrow \qquad n = 7$$

Step 4: Verify.

$$S_7 = \frac{4\left(3^7 - 1\right)}{2} = \frac{4 \times 2186}{2} = 2 \times 2186 = 4372\checkmark$$

$$\boxed{n = 7 \text{ terms}}$$

8. Sum $1 + 2 + 4 + \cdots + 512$
Step 1: Identify t_1 and r.

$$t_1 = 1 \qquad r = 2$$

Step 2: Find n by setting $t_n = 512$.

$$1 \cdot 2^{n-1} = 512 = 2^9 \qquad \Rightarrow \qquad n - 1 = 9 \qquad \Rightarrow \qquad n = 10$$

Step 3: Apply the formula.

$$S_{10} = \frac{1\,(2^{10} - 1)}{2 - 1} = 2^{10} - 1 = 1024 - 1$$

$$\boxed{S_{10} = 1023}$$

9. $t_1 = 192$, $r = \dfrac{1}{2}$, $n = 5$
Step 1: Use the alternate form since $r < 1$.

$$S_5 = \frac{192\left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}}$$

Step 2: Evaluate $\left(\frac{1}{2}\right)^5 = \dfrac{1}{32}$.

$$S_5 = \frac{192\left(1 - \dfrac{1}{32}\right)}{\dfrac{1}{2}} = \frac{192 \times \dfrac{31}{32}}{\dfrac{1}{2}}$$

Step 3: Multiply by 2 (dividing by $\frac{1}{2}$).

$$S_5 = 192 \times \frac{31}{32} \times 2 = \frac{192 \times 31 \times 2}{32} = 6 \times 31 \times 2 = 12 \times 31$$

$$\boxed{S_5 = 372}$$

(Quick check: $192 + 96 + 48 + 24 + 12 = 372$ ✓)

10. (Challenge) Ball dropped from 32 m, each bounce rises to $\frac{1}{2}$ the previous height, 4 bounces total.

Step 1: Understand the structure of the journey.

The ball falls 32 m on the way down. Then for each bounce it travels *up* and *back down* the same height. The four bounce heights form a geometric sequence with $t_1 = 32 \times \frac{1}{2} = 16$ m and $r = \frac{1}{2}$.

Step 2: Find the total height covered in the four bounces (up + down for each).

Each bounce contributes $2 \times$ (bounce height) to the total distance. So we need $2 \times S_4$ of the bounce heights.

$$S_4 = \frac{16\left(1 - \left(\frac{1}{2}\right)^4\right)}{1 - \frac{1}{2}} = \frac{16\left(1 - \dfrac{1}{16}\right)}{\dfrac{1}{2}} = \frac{16 \times \dfrac{15}{16}}{\dfrac{1}{2}} = 15 \times 2 = 30$$

Step 3: Add the initial drop.

$$\text{Total distance} = \underbrace{32}_{\text{initial drop}} + \underbrace{2 \times 30}_{\text{4 bounces up and down}} = 32 + 60$$

$$\boxed{\text{Total distance} = 92 \text{ m}}$$

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