Infinite Geometric Series
ApexMath

Notes

What Is an Infinite Geometric Series?

In Lesson 5 we found the sum of the first n terms of a geometric series. But what happens if we keep adding terms *forever*? That is an **infinite geometric series**:

$$t_1 + t_1 r + t_1 r^2 + t_1 r^3 + \cdots$$

The answer depends entirely on the value of r.

When Does an Infinite Series Have a Finite Sum?

Consider the sum formula from Lesson 5:

$$S_n = \frac{t_1(1 - r^n)}{1 - r}$$

Ask: what happens to r^n as n gets larger and larger?

- If $|r| < 1$ (i.e. $-1 < r < 1$), then r^n gets closer and closer to zero as $n \to \infty$. The sum **converges** to a finite value.
- If $|r| \geq 1$, then r^n does not shrink. The partial sums grow without bound and the series **diverges** — there is no finite sum.

The Infinite Sum Formula

When $|r| < 1$, substituting $r^n \to 0$ into the partial sum formula gives:

$$S_\infty = \frac{t_1(1 - 0)}{1 - r}$$

$$\boxed{S_\infty = \frac{t_1}{1 - r}, \qquad \text{provided } |r| < 1}$$

This is the only formula needed for this entire lesson. Memorise it, and always check the condition $|r| < 1$ before using it.

Summary of Convergence

- $|r| < 1$: series **converges**. Use $S_\infty = \dfrac{t_1}{1 - r}$.
- $|r| \geq 1$: series **diverges**. There is no finite sum.

Finding t_1 or r When S_∞ Is Given
Rearranging $S_\infty = \dfrac{t_1}{1-r}$ gives:

$$t_1 = S_\infty\,(1-r) \qquad \text{or} \qquad r = 1 - \frac{t_1}{S_\infty}$$

Use these rearrangements when one piece of information is missing and the sum is given.

Repeating Decimals as Infinite Series
Every repeating decimal is an infinite geometric series in disguise. For example:

$$0.\overline{9} = 0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

This is a geometric series with $t_1 = \dfrac{9}{10}$ and $r = \dfrac{1}{10}$, so:

$$S_\infty = \frac{9/10}{1 - 1/10} = \frac{9/10}{9/10} = 1$$

This proves that $0.\overline{9} = 1$ exactly.

Pro Tip

- **Always check $|r| < 1$ first.** If this condition is not met, stop immediately — there is no finite sum and the problem is done.
- **Identify t_1 carefully from the series.** t_1 is the very first term, not the coefficient in the general term formula. For example, in $4 - 2 + 1 - \frac{1}{2} + \cdots$, we have $t_1 = 4$ and $r = -\frac{1}{2}$.
- **When r is negative, the series alternates in sign but can still converge.** As long as $|r| < 1$, the formula works perfectly.
- **To convert a repeating decimal, identify the repeating block, write it as a fraction over the appropriate power of 10, and use that as t_1.**
- **A fractional answer is completely normal here.** Many infinite sums produce fractions. Do not convert to a decimal unless specifically asked.

Examples

Example 1 (Basic application): Find S_∞ for the geometric series with $t_1 = 8$ and $r = \frac{1}{2}$.

Step 1: Check the convergence condition.

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1 \qquad \checkmark \text{ The series converges.}$$

Step 2: Apply the formula.

$$S_\infty = \frac{t_1}{1-r} = \frac{8}{1-\frac{1}{2}} = \frac{8}{\frac{1}{2}}$$

Step 3: Dividing by $\frac{1}{2}$ is the same as multiplying by 2.

$$S_\infty = 8 \times 2$$

$$\boxed{S_\infty = 16}$$

Example 2 (Negative common ratio): Find S_∞ for the geometric series with $t_1 = 15$ and $r = -\frac{1}{3}$.

Step 1: Check the convergence condition.

$$\left|-\frac{1}{3}\right| = \frac{1}{3} < 1 \qquad \checkmark \text{ The series converges.}$$

Step 2: Apply the formula.

$$S_\infty = \frac{15}{1-\left(-\frac{1}{3}\right)} = \frac{15}{1+\frac{1}{3}} = \frac{15}{\frac{4}{3}}$$

Step 3: Dividing by $\frac{4}{3}$ is multiplying by $\frac{3}{4}$.

$$S_\infty = 15 \times \frac{3}{4} = \frac{45}{4}$$

$$\boxed{S_\infty = \frac{45}{4}}$$

Example 3 (Repeating decimal): Prove that $0.\overline{9} = 1$ using the infinite series formula.

Step 1: Write the decimal as a geometric series.

$$0.\overline{9} = 0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$t_1 = \frac{9}{10} \qquad r = \frac{1}{10}$$

Step 2: Check $\left|\frac{1}{10}\right| = \frac{1}{10} < 1$**.** ✓

Step 3: Apply the formula.

$$S_\infty = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}}$$

$$\boxed{S_\infty = 1}$$

Therefore $0.\overline{9} = 1$ exactly.

Example 4 (Larger common ratio): Find S_∞ for the geometric series with $t_1 = 12$ and $r = \frac{2}{3}$.

Step 1: Check $\left|\frac{2}{3}\right| = \frac{2}{3} < 1$**.** ✓

Step 2: Apply the formula.

$$S_\infty = \frac{12}{1 - \frac{2}{3}} = \frac{12}{\frac{1}{3}} = 12 \times 3$$

$$\boxed{S_\infty = 36}$$

Example 5 (Finding t_1 when S_∞ is given): An infinite geometric series has $S_\infty = 20$ and $r = \frac{3}{5}$. Find t_1.

Step 1: Check $\left|\frac{3}{5}\right| = \frac{3}{5} < 1$**.** ✓

Step 2: Rearrange the formula to solve for t_1.

$$S_\infty = \frac{t_1}{1-r} \qquad \Rightarrow \qquad t_1 = S_\infty\,(1-r)$$

Step 3: Substitute the known values.

$$t_1 = 20\left(1 - \frac{3}{5}\right) = 20 \times \frac{2}{5} = \frac{40}{5}$$

$$\boxed{t_1 = 8}$$

Example 6 (Divergent series): Does the series with $t_1 = 5$ and $r = 2$ have a finite sum?

Step 1: Check the convergence condition.

$$|r| = |2| = 2 \geq 1$$

The condition $|r| < 1$ is **not** satisfied.

$$\boxed{\text{The series diverges. There is no finite sum.}}$$

Common Mistakes

- **Forgetting to check $|r| < 1$ before applying the formula.** If $|r| \geq 1$, the formula gives a wrong or meaningless answer. Always check first.
- **Writing $1 + r$ instead of $1 - r$ in the denominator.** The denominator is always $(1 - r)$. When r is negative, this becomes $1 - (-r) = 1 + |r|$, which is larger than 1.
- **Confusing t_1 with a term coefficient.** Make sure t_1 is the actual numerical value of the first term, not a symbol.
- **Thinking r must be positive.** Negative values of r are perfectly valid as long as $|r| < 1$.
- **Rounding the answer to a decimal when a fraction is exact.** An answer like $\frac{45}{4}$ is exact. Unless a decimal is requested, leave it as a fraction.

Worksheet

1. Find S_∞ for the geometric series with $t_1 = 6$ and $r = \frac{1}{2}$.

2. Find S_∞ for the geometric series with $t_1 = 10$ and $r = \frac{1}{4}$.

3. Find S_∞ for the geometric series with $t_1 = 9$ and $r = -\frac{1}{3}$.

4. Find S_∞ for the geometric series with $t_1 = 100$ and $r = \frac{4}{5}$.

5. Find S_∞ for the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

6. Find S_∞ for the series $4 - 2 + 1 - \frac{1}{2} + \cdots$

(Identify r first by dividing consecutive terms.)

7. Does the series with $t_1 = 3$ and $r = 5$ have a finite sum? Justify your answer.

8. An infinite geometric series has $S_\infty = 25$ and $r = \dfrac{1}{5}$. Find t_1.

9. An infinite geometric series has $S_\infty = 18$ and $r = \dfrac{2}{3}$. Find t_1.

10. An infinite geometric series has $t_1 = 5$ and $S_\infty = 20$. Find r.

11. *(Challenge)* Express $0.\overline{36}$ as a fraction in lowest terms using the infinite series method. *(Hint:* $0.\overline{36} = 0.363636\ldots$ *Write this as a geometric series with* $t_1 = \dfrac{36}{100}$.*)*

12. *(Challenge)* An infinite geometric series has $t_1 = 8$ and $S_\infty = 32$. Find r and verify your answer.

Answer Key

1. $t_1 = 6$, $r = \dfrac{1}{2}$

Step 1: Check $\left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$**.** ✓

Step 2: Apply the formula.

$$S_\infty = \frac{6}{1 - \frac{1}{2}} = \frac{6}{\frac{1}{2}} = 6 \times 2$$

$$\boxed{S_\infty = 12}$$

2. $t_1 = 10$, $r = \dfrac{1}{4}$

Step 1: Check $\left|\dfrac{1}{4}\right| = \dfrac{1}{4} < 1$**.** ✓

Step 2: Apply the formula.

$$S_\infty = \frac{10}{1 - \frac{1}{4}} = \frac{10}{\frac{3}{4}} = 10 \times \frac{4}{3} = \frac{40}{3}$$

$$\boxed{S_\infty = \frac{40}{3}}$$

3. $t_1 = 9$, $r = -\dfrac{1}{3}$

Step 1: Check $\left|-\dfrac{1}{3}\right| = \dfrac{1}{3} < 1$**.** ✓

Step 2: Apply the formula. Be careful with the negative r**.**

$$S_\infty = \frac{9}{1 - \left(-\frac{1}{3}\right)} = \frac{9}{1 + \frac{1}{3}} = \frac{9}{\frac{4}{3}} = 9 \times \frac{3}{4} = \frac{27}{4}$$

$$\boxed{S_\infty = \frac{27}{4}}$$

4. $t_1 = 100$, $r = \dfrac{4}{5}$

Step 1: Check $\left|\dfrac{4}{5}\right| = \dfrac{4}{5} < 1$. ✓

Step 2: Apply the formula.

$$S_\infty = \frac{100}{1 - \frac{4}{5}} = \frac{100}{\frac{1}{5}} = 100 \times 5$$

$$\boxed{S_\infty = 500}$$

5. Series: $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots$

Step 1: Identify t_1 and r.

$$t_1 = 1 \qquad r = \frac{1/3}{1} = \frac{1}{3}$$

Step 2: Check $\left|\dfrac{1}{3}\right| = \dfrac{1}{3} < 1$. ✓

Step 3: Apply the formula.

$$S_\infty = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = 1 \times \frac{3}{2}$$

$$\boxed{S_\infty = \frac{3}{2}}$$

6. Series: $4 - 2 + 1 - \dfrac{1}{2} + \cdots$

Step 1: Identify t_1 and r.

$$t_1 = 4 \qquad r = \frac{-2}{4} = -\frac{1}{2}$$

Step 2: Check $\left|-\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$. ✓

Step 3: Apply the formula.

$$S_\infty = \frac{4}{1 - \left(-\frac{1}{2}\right)} = \frac{4}{1 + \frac{1}{2}} = \frac{4}{\frac{3}{2}} = 4 \times \frac{2}{3} = \frac{8}{3}$$

$$\boxed{S_\infty = \frac{8}{3}}$$

7. $t_1 = 3$, $r = 5$
Step 1: Check the convergence condition.

$$|r| = |5| = 5 \geq 1$$

The condition $|r| < 1$ is not satisfied.

$$\boxed{\text{The series } \textbf{diverges}. \text{ There is no finite sum.}}$$

8. $S_\infty = 25$, $r = \dfrac{1}{5}$. Find t_1.
Step 1: Check $\left|\dfrac{1}{5}\right| = \dfrac{1}{5} < 1$**.** ✓
Step 2: Rearrange to solve for t_1**.**

$$t_1 = S_\infty (1 - r) = 25\left(1 - \frac{1}{5}\right) = 25 \times \frac{4}{5} = \frac{100}{5}$$

$$\boxed{t_1 = 20}$$

9. $S_\infty = 18$, $r = \dfrac{2}{3}$. Find t_1.
Step 1: Check $\left|\dfrac{2}{3}\right| = \dfrac{2}{3} < 1$**.** ✓
Step 2: Rearrange to solve for t_1**.**

$$t_1 = S_\infty (1 - r) = 18\left(1 - \frac{2}{3}\right) = 18 \times \frac{1}{3} = \frac{18}{3}$$

$$\boxed{t_1 = 6}$$

10. $t_1 = 5$, $S_\infty = 20$. Find r.
Step 1: Rearrange the formula to isolate r.

$$S_\infty = \frac{t_1}{1-r} \qquad \Rightarrow \qquad 1 - r = \frac{t_1}{S_\infty}$$

Step 2: Substitute the known values.

$$1 - r = \frac{5}{20} = \frac{1}{4}$$

Step 3: Solve for r.

$$r = 1 - \frac{1}{4} = \frac{3}{4}$$

Step 4: Check $\left|\frac{3}{4}\right| < 1$. ✓ Verify.

$$S_\infty = \frac{5}{1 - \frac{3}{4}} = \frac{5}{\frac{1}{4}} = 5 \times 4 = 20\checkmark$$

$$\boxed{r = \frac{3}{4}}$$

11. (Challenge) Express $0.\overline{36}$ as a fraction in lowest terms.
Step 1: Write the repeating decimal as a geometric series.

$$0.\overline{36} = 0.363636\ldots = \frac{36}{100} + \frac{36}{10000} + \frac{36}{1000000} + \cdots$$

Step 2: Identify t_1 and r.

$$t_1 = \frac{36}{100} \qquad r = \frac{1}{100}$$

Step 3: Check $\left|\frac{1}{100}\right| < 1$. ✓
Step 4: Apply the formula.

$$S_\infty = \frac{\frac{36}{100}}{1 - \frac{1}{100}} = \frac{\frac{36}{100}}{\frac{99}{100}} = \frac{36}{100} \times \frac{100}{99} = \frac{36}{99}$$

Step 5: Simplify. The GCD of 36 and 99 is 9.

$$\frac{36}{99} = \frac{36 \div 9}{99 \div 9} = \frac{4}{11}$$

$$\boxed{0.\overline{36} = \frac{4}{11}}$$

12. (Challenge) $t_1 = 8$, $S_\infty = 32$. Find r.

Step 1: Rearrange to isolate r.

$$1 - r = \frac{t_1}{S_\infty} = \frac{8}{32} = \frac{1}{4}$$

Step 2: Solve for r.

$$r = 1 - \frac{1}{4} = \frac{3}{4}$$

Step 3: Check $\left|\frac{3}{4}\right| = \frac{3}{4} < 1$**.** ✓

Step 4: Verify.

$$S_\infty = \frac{8}{1 - \frac{3}{4}} = \frac{8}{\frac{1}{4}} = 8 \times 4 = 32\checkmark$$

$$\boxed{r = \frac{3}{4}}$$

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