Introduction to Sequences
ApexMath

Notes

What Is a Sequence?

A **sequence** is an ordered list of numbers. Each number in the list is called a **term**. The position of a term in the list is called its **index**.

For example:

$$3,\ 7,\ 11,\ 15,\ 19,\ \ldots$$

The first term is 3, the second term is 7, the third term is 11, and so on. The three dots $(\ldots)$ mean the list continues following the same pattern.

Notation

We use the letter t with a subscript to refer to specific terms:

- t_1 means the **first** term
- t_2 means the **second** term
- t_n means the n**th term** (the term in position n)

So for the sequence $3,\ 7,\ 11,\ 15,\ 19,\ \ldots$, we write: $t_1 = 3,\ t_2 = 7,\ t_3 = 11,\ t_4 = 15,\ t_5 = 19$.

Two Types of Sequences

In this unit we study two specific types of sequences.

Type 1 — Arithmetic Sequences

An **arithmetic sequence** is one where you *add* the same number each time to get to the next term. That fixed number is called the **common difference**, written d.

$$\text{Common difference: } d = t_2 - t_1 = t_3 - t_2 = \cdots$$

Example: $3,\ 7,\ 11,\ 15,\ 19,\ \ldots$

Here $d = 7 - 3 = 4$. Each term is found by adding 4 to the previous one.

Type 2 — Geometric Sequences

A **geometric sequence** is one where you *multiply* by the same number each time. That fixed number is called the **common ratio**, written r.

$$\text{Common ratio: } r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots$$

Example: $2,\ 6,\ 18,\ 54,\ 162,\ \ldots$

Here $r = \dfrac{6}{2} = 3$. Each term is found by multiplying the previous term by 3.

The General Term Formula

Instead of listing every term one by one, we use a formula for t_n. This lets us jump directly to any term — for example, the 50th term — without computing all the ones in between.

For an **arithmetic** sequence with first term t_1 and common difference d:

$$\boxed{t_n = t_1 + (n-1)\,d}$$

For a **geometric** sequence with first term t_1 and common ratio r:

$$\boxed{t_n = t_1 \cdot r^{\,n-1}}$$

Where Do These Formulas Come From?

For the arithmetic formula, think about it step by step. To reach the nth term starting from t_1, you add d exactly $(n-1)$ times:

$$t_n = \underbrace{t_1}_{\text{start}} + \underbrace{(n-1)\cdot d}_{\text{add } d \text{ a total of } (n-1) \text{ times}}$$

For the geometric formula, to reach the nth term you multiply by r exactly $(n-1)$ times:

$$t_n = \underbrace{t_1}_{\text{start}} \times \underbrace{r^{\,n-1}}_{\text{multiply by } r \text{ a total of } (n-1) \text{ times}}$$

It is always $(n-1)$, not n, because you do not multiply when moving from nothing to the first term — you already start there.

General Term from a Formula

Sometimes a sequence is given to you as a formula. For example:

$$t_n = 4n - 1$$

To list terms, substitute $n = 1, 2, 3, \ldots$ into the formula. To find a specific term, substitute that value of n directly.

Pro Tip

- **Check arithmetic by subtracting.** To confirm a sequence is arithmetic, subtract any term from the next one. If you get the same number every time, the common difference is d.
- **Check geometric by dividing.** To confirm a sequence is geometric, divide any term by the one before it. If you get the same number every time, the common ratio is r.
- **Always identify t_1, d (or r), and the target n before using any formula.** Write these down before substituting. Students who skip this step often substitute the wrong values.

- **The exponent is $n-1$, not n.** In $t_n = t_1 \cdot r^{n-1}$, the power is $(n-1)$. Substituting n instead of $(n-1)$ is the single most common error in this topic.
- **d can be negative.** If a sequence is decreasing, the common difference is a negative number. For example, in $20, 17, 14, 11, \ldots$, the common difference is $d = -3$.
- **r can be a fraction.** Geometric sequences can decrease. In $81, 27, 9, 3, \ldots$, the common ratio is $r = \frac{1}{3}$.

Examples

Example 1 (Listing terms from t_1 and d): Write the first five terms of the arithmetic sequence with $t_1 = 3$ and $d = 4$.

Step 1: Start with the first term.

$$t_1 = 3$$

Step 2: Add $d = 4$ each time to get the next term.

$$t_2 = 3 + 4 = 7$$
$$t_3 = 7 + 4 = 11$$
$$t_4 = 11 + 4 = 15$$
$$t_5 = 15 + 4 = 19$$

The first five terms are:

$$\boxed{3,\ 7,\ 11,\ 15,\ 19}$$

Example 2 (Listing terms from t_1 and r): Write the first five terms of the geometric sequence with $t_1 = 2$ and $r = 3$.

Step 1: Start with the first term.

$$t_1 = 2$$

Step 2: Multiply by $r = 3$ each time.

$$t_2 = 2 \times 3 = 6$$
$$t_3 = 6 \times 3 = 18$$
$$t_4 = 18 \times 3 = 54$$
$$t_5 = 54 \times 3 = 162$$

The first five terms are:

$$\boxed{2,\ 6,\ 18,\ 54,\ 162}$$

Example 3 (Finding a specific term — Arithmetic): The sequence $5,\ 8,\ 11,\ 14,\ \ldots$ is arithmetic. Find the 20th term, t_{20}.

Step 1: Identify t_1, d, and n.
The first term is $t_1 = 5$. The common difference is $d = 8 - 5 = 3$. The target term is $n = 20$.
Step 2: Write the formula.

$$t_n = t_1 + (n-1)\,d$$

Step 3: Substitute the known values.

$$t_{20} = 5 + (20-1) \times 3$$

Step 4: Simplify inside the brackets first.

$$t_{20} = 5 + 19 \times 3$$

$$t_{20} = 5 + 57$$

$$\boxed{t_{20} = 62}$$

Example 4 (Finding a specific term — Geometric): The sequence $6,\ 18,\ 54,\ \ldots$ is geometric. Find the 6th term, t_6.

Step 1: Identify t_1, r, and n.
The first term is $t_1 = 6$. The common ratio is $r = \dfrac{18}{6} = 3$. The target term is $n = 6$.
Step 2: Write the formula.

$$t_n = t_1 \cdot r^{\,n-1}$$

Step 3: Substitute the known values.

$$t_6 = 6 \times 3^{\,6-1}$$

Step 4: Evaluate the exponent, then multiply.

$$t_6 = 6 \times 3^5 = 6 \times 243$$

$$\boxed{t_6 = 1458}$$

Example 5 (Using a given formula — Arithmetic): A sequence is defined by $t_n = 4n - 1$. List the first four terms and identify the type of sequence.

Step 1: Substitute $n = 1, 2, 3, 4$ into the formula.

$$t_1 = 4(1) - 1 = 3$$
$$t_2 = 4(2) - 1 = 7$$
$$t_3 = 4(3) - 1 = 11$$
$$t_4 = 4(4) - 1 = 15$$

Step 2: Check whether it is arithmetic or geometric.
Check the differences: $7 - 3 = 4,\ 11 - 7 = 4,\ 15 - 11 = 4$.
The differences are constant, so this is an **arithmetic** sequence with $d = 4$.

$$\boxed{3,\ 7,\ 11,\ 15 \text{ — arithmetic, } d = 4}$$

Example 6 (Using a given formula — Geometric): A sequence is defined by $t_n = 3 \cdot 2^{n-1}$. List the first four terms and identify the type.

Step 1: Substitute $n = 1, 2, 3, 4$.

$$t_1 = 3 \times 2^0 = 3 \times 1 = 3$$
$$t_2 = 3 \times 2^1 = 3 \times 2 = 6$$
$$t_3 = 3 \times 2^2 = 3 \times 4 = 12$$
$$t_4 = 3 \times 2^3 = 3 \times 8 = 24$$

Step 2: Check the ratios.
$\frac{6}{3} = 2,\ \frac{12}{6} = 2,\ \frac{24}{12} = 2$. The ratio is constant, so this is a **geometric** sequence with $r = 2$.

$$\boxed{3,\ 6,\ 12,\ 24 \text{ — geometric, } r = 2}$$

Common Mistakes

- **Using r^n instead of r^{n-1}** in the geometric formula. The exponent is always $(n - 1)$ because you start at t_1 and multiply $(n - 1)$ times to reach t_n.
- **Forgetting that d or r can be negative or fractional.** Always subtract (or divide) consecutive terms carefully — do not assume the values are positive whole numbers.
- **Confusing arithmetic and geometric.** Arithmetic sequences have a constant *difference*; geometric sequences have a constant *ratio*. Check both before deciding.

- **Misidentifying t_1.** In the formula, t_1 is the very first term of the sequence, not the term at $n = 0$. Sequences in this course start at $n = 1$.
- **Computing r by subtracting instead of dividing.** For geometric sequences, the ratio is found by *dividing* $t_2 \div t_1$, not subtracting $t_2 - t_1$.

Worksheet

1. The sequence $4,\ 7,\ 10,\ 13,\ \ldots$ is arithmetic.
   (a) Write the next two terms.
   (b) State the values of t_1 and d.
   (c) Find t_{15}.
   (d) Write a general formula for t_n.

2. The sequence $5,\ 10,\ 20,\ 40,\ \ldots$ is geometric.
   (a) Write the next two terms.
   (b) State the values of t_1 and r.
   (c) Find t_8.
   (d) Write a general formula for t_n.

3. The sequence $20,\ 17,\ 14,\ 11,\ \ldots$ is arithmetic.
   (a) Write the next two terms.
   (b) State the values of t_1 and d.
   (c) Find t_{12}.
   (d) Write a general formula for t_n.

4. The sequence $81,\ 27,\ 9,\ 3,\ \ldots$ is geometric.
   (a) Write the next two terms.
   (b) State the values of t_1 and r.
   (c) Find t_7.
   (d) Write a general formula for t_n.

5. A sequence is defined by the formula $t_n = 5n + 2$.
   (a) List the first five terms.
   (b) Is this sequence arithmetic or geometric? Justify your answer.
   (c) Find t_{20}.

6. A sequence is defined by the formula $t_n = 4 \cdot 3^{n-1}$.
   (a) List the first four terms.
   (b) Is this sequence arithmetic or geometric? Justify your answer.
   (c) Find t_5.

7. An arithmetic sequence has $t_3 = 11$ and $t_7 = 27$. Find t_1, d, and t_{10}.

8. A geometric sequence has $t_2 = 6$ and $t_5 = 162$. Find t_1, r, and t_6.

9. Classify each sequence below as arithmetic, geometric, or neither. If arithmetic, state d. If geometric, state r.

   (a) $1,\ 4,\ 9,\ 16,\ 25,\ \ldots$

   (b) $2,\ 6,\ 18,\ 54,\ \ldots$

10. *(Challenge)* The sequence $4,\ 7,\ 10,\ \ldots$ is arithmetic with $t_1 = 4$ and $d = 3$. Is 100 a term in this sequence? If so, which term is it?

Answer Key

1. Sequence: $4,\ 7,\ 10,\ 13,\ \ldots$

(a) Next two terms.
The common difference is $d = 7 - 4 = 3$. Add 3 to get each new term:

$$t_5 = 13 + 3 = 16 \qquad t_6 = 16 + 3 = 19$$

$$\boxed{\text{Next two terms: 16 and 19}}$$

(b) State t_1 and d.
The sequence starts at 4 and each term increases by 3:

$$\boxed{t_1 = 4, \quad d = 3}$$

(c) Find t_{15}.
Step 1: Write the formula.

$$t_n = t_1 + (n-1)\,d$$

Step 2: Substitute $t_1 = 4$, $d = 3$, $n = 15$.

$$t_{15} = 4 + (15 - 1) \times 3$$

Step 3: Simplify.

$$t_{15} = 4 + 14 \times 3 = 4 + 42$$

$$\boxed{t_{15} = 46}$$

(d) General formula for t_n.
Substituting $t_1 = 4$ and $d = 3$ into $t_n = t_1 + (n-1)d$:

$$t_n = 4 + (n-1)(3) = 4 + 3n - 3 = 3n + 1$$

$$\boxed{t_n = 3n + 1}$$

2. Sequence: 5, 10, 20, 40, …

(a) Next two terms.

The common ratio is $r = \dfrac{10}{5} = 2$. Multiply by 2 each time:

$$t_5 = 40 \times 2 = 80 \qquad t_6 = 80 \times 2 = 160$$

$$\boxed{\text{Next two terms: 80 and 160}}$$

(b) State t_1 and r.

$$\boxed{t_1 = 5, \quad r = 2}$$

(c) Find t_8.

Step 1: Write the formula.

$$t_n = t_1 \cdot r^{\,n-1}$$

Step 2: Substitute $t_1 = 5$, $r = 2$, $n = 8$.

$$t_8 = 5 \times 2^{\,8-1} = 5 \times 2^7$$

Step 3: Evaluate.

$$2^7 = 128 \qquad \Rightarrow \qquad t_8 = 5 \times 128$$

$$\boxed{t_8 = 640}$$

(d) General formula for t_n.

$$\boxed{t_n = 5 \cdot 2^{\,n-1}}$$

3. Sequence: $20,\ 17,\ 14,\ 11,\ \ldots$

(a) Next two terms.

The common difference is $d = 17 - 20 = -3$. Add -3 (i.e. subtract 3) each time:

$$t_5 = 11 + (-3) = 8 \qquad t_6 = 8 + (-3) = 5$$

$$\boxed{\text{Next two terms: 8 and 5}}$$

(b) State t_1 and d.

$$\boxed{t_1 = 20, \quad d = -3}$$

(c) Find t_{12}.

Step 1: Write the formula.

$$t_n = t_1 + (n-1)\,d$$

Step 2: Substitute $t_1 = 20$, $d = -3$, $n = 12$.

$$t_{12} = 20 + (12 - 1) \times (-3)$$

Step 3: Simplify.

$$t_{12} = 20 + 11 \times (-3) = 20 + (-33) = 20 - 33$$

$$\boxed{t_{12} = -13}$$

(d) General formula for t_n.

$$t_n = 20 + (n-1)(-3) = 20 - 3n + 3 = 23 - 3n$$

$$\boxed{t_n = 23 - 3n}$$

4. Sequence: $81,\ 27,\ 9,\ 3,\ \ldots$

(a) Next two terms.

The common ratio is $r = \dfrac{27}{81} = \dfrac{1}{3}$. Multiply by $\dfrac{1}{3}$ each time:

$$t_5 = 3 \times \frac{1}{3} = 1 \qquad t_6 = 1 \times \frac{1}{3} = \frac{1}{3}$$

$$\boxed{\text{Next two terms: } 1 \text{ and } \frac{1}{3}}$$

(b) State t_1 and r.

$$\boxed{t_1 = 81, \quad r = \frac{1}{3}}$$

(c) Find t_7.

Step 1: Write the formula.

$$t_n = t_1 \cdot r^{n-1}$$

Step 2: Substitute $t_1 = 81$, $r = \dfrac{1}{3}$, $n = 7$.

$$t_7 = 81 \times \left(\frac{1}{3}\right)^6$$

Step 3: Evaluate. Note that $\left(\dfrac{1}{3}\right)^6 = \dfrac{1}{729}$.

$$t_7 = 81 \times \frac{1}{729} = \frac{81}{729} = \frac{1}{9}$$

$$\boxed{t_7 = \frac{1}{9}}$$

(d) General formula for t_n.

$$\boxed{t_n = 81 \cdot \left(\frac{1}{3}\right)^{n-1}}$$

5. $t_n = 5n + 2$

(a) First five terms.

Substitute $n = 1, 2, 3, 4, 5$ into $t_n = 5n + 2$:

$$\begin{aligned} t_1 &= 5(1) + 2 = 7 \\ t_2 &= 5(2) + 2 = 12 \\ t_3 &= 5(3) + 2 = 17 \\ t_4 &= 5(4) + 2 = 22 \\ t_5 &= 5(5) + 2 = 27 \end{aligned}$$

$$\boxed{\text{First five terms: } 7,\ 12,\ 17,\ 22,\ 27}$$

(b) Arithmetic or geometric?

Check the differences: $12 - 7 = 5,\ \ 17 - 12 = 5,\ \ 22 - 17 = 5$. The difference is constant at 5, so this is an **arithmetic sequence** with $d = 5$.

(c) Find t_{20}.

$$t_{20} = 5(20) + 2 = 100 + 2$$

$$\boxed{t_{20} = 102}$$

6. $t_n = 4 \cdot 3^{n-1}$

(a) First four terms.

Substitute $n = 1, 2, 3, 4$:

$$\begin{aligned} t_1 &= 4 \times 3^0 = 4 \times 1 = 4 \\ t_2 &= 4 \times 3^1 = 4 \times 3 = 12 \\ t_3 &= 4 \times 3^2 = 4 \times 9 = 36 \\ t_4 &= 4 \times 3^3 = 4 \times 27 = 108 \end{aligned}$$

$$\boxed{\text{First four terms: } 4,\ 12,\ 36,\ 108}$$

(b) Arithmetic or geometric?

Check the ratios: $\dfrac{12}{4} = 3,\ \ \dfrac{36}{12} = 3,\ \ \dfrac{108}{36} = 3$. The ratio is constant at 3, so this is a **geometric sequence** with $r = 3$.

(c) Find t_5.

$$t_5 = 4 \times 3^{5-1} = 4 \times 3^4 = 4 \times 81$$

$$\boxed{t_5 = 324}$$

7. Arithmetic sequence with $t_3 = 11$ and $t_7 = 27$.
Step 1: Set up two equations using the formula $t_n = t_1 + (n-1)d$.
For $t_3 = 11$:

$$t_1 + 2d = 11 \qquad \cdots\ (1)$$

For $t_7 = 27$:

$$t_1 + 6d = 27 \qquad \cdots\ (2)$$

Step 2: Subtract equation (1) from equation (2) to eliminate t_1.

$$(t_1 + 6d) - (t_1 + 2d) = 27 - 11$$
$$4d = 16$$
$$d = 4$$

Step 3: Substitute $d = 4$ back into equation (1) to find t_1.

$$t_1 + 2(4) = 11$$
$$t_1 + 8 = 11$$
$$t_1 = 3$$

Step 4: Find t_{10} using the formula.

$$t_{10} = t_1 + (10-1)d = 3 + 9 \times 4 = 3 + 36$$

$$\boxed{t_1 = 3, \quad d = 4, \quad t_{10} = 39}$$

8. Geometric sequence with $t_2 = 6$ and $t_5 = 162$.
Step 1: Write two equations using $t_n = t_1 \cdot r^{n-1}$.
For $t_2 = 6$:

$$t_1 \cdot r = 6 \qquad \cdots\ (1)$$

For $t_5 = 162$:

$$t_1 \cdot r^4 = 162 \qquad \cdots\ (2)$$

Step 2: Divide equation (2) by equation (1) to eliminate t_1.

$$\frac{t_1 \cdot r^4}{t_1 \cdot r} = \frac{162}{6}$$

$$r^3 = 27$$

$$r = 3$$

Step 3: Substitute $r = 3$ back into equation (1) to find t_1.

$$t_1 \times 3 = 6$$

$$t_1 = 2$$

Step 4: Find t_6.

$$t_6 = 2 \times 3^{6-1} = 2 \times 3^5 = 2 \times 243$$

$$\boxed{t_1 = 2, \quad r = 3, \quad t_6 = 486}$$

9.
(a) Sequence: $1,\ 4,\ 9,\ 16,\ 25,\ \ldots$
Step 1: Check for arithmetic (constant differences).

$$4 - 1 = 3, \qquad 9 - 4 = 5, \qquad 16 - 9 = 7$$

The differences are not constant, so it is **not arithmetic**.
Step 2: Check for geometric (constant ratios).

$$\frac{4}{1} = 4, \qquad \frac{9}{4} = 2.25, \qquad \frac{16}{9} \approx 1.78$$

The ratios are not constant, so it is **not geometric**.

$$\boxed{\text{Neither. These are perfect squares: } t_n = n^2}$$

(b) Sequence: $2,\ 6,\ 18,\ 54,\ \ldots$
Step 1: Check for arithmetic.

$$6 - 2 = 4, \qquad 18 - 6 = 12$$

Not constant, so not arithmetic.

Step 2: Check for geometric.

$$\frac{6}{2} = 3, \qquad \frac{18}{6} = 3, \qquad \frac{54}{18} = 3$$

The ratio is constant at 3.

$$\boxed{\text{Geometric with } r = 3.\ \text{ General term: } t_n = 2 \cdot 3^{n-1}}$$

10. (Challenge) Is 100 a term in $4,\ 7,\ 10,\ \ldots$?

Step 1: Identify t_1 and d.

$$t_1 = 4, \qquad d = 3$$

Step 2: Set the general term equal to 100 and solve for n.

$$t_n = t_1 + (n-1)\,d$$
$$100 = 4 + (n-1) \times 3$$

Step 3: Isolate $(n-1)$.

$$100 - 4 = (n-1) \times 3$$
$$96 = (n-1) \times 3$$
$$n - 1 = \frac{96}{3} = 32$$
$$n = 33$$

Step 4: Since $n = 33$ is a positive whole number, 100 is indeed a term.

Verify: $t_{33} = 4 + (33-1)(3) = 4 + 96 = 100$. ✓

$$\boxed{\text{100 is the 33rd term of the sequence.}}$$

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