Sigma Notation
ApexMath

Notes

What Is Sigma Notation?

In earlier lessons we wrote series by listing terms:

$$3 + 7 + 11 + 15 + 19$$

This works for short sums, but becomes impractical for long ones. **Sigma notation** (also called **summation notation**) gives us a compact way to write any sum. It uses the Greek capital letter sigma, Σ, which stands for "sum".

Reading Sigma Notation

A sigma expression looks like this:

$$\sum_{k=1}^{n} f(k)$$

Here is what each part means:

- Σ — the instruction to add up all values
- k — the **index** (the counter variable, sometimes written i or j)
- $k = 1$ (below Σ) — the **lower limit**: the starting value of k
- n (above Σ) — the **upper limit**: the ending value of k
- $f(k)$ — the **general term**: the expression to evaluate at each value of k

How to Evaluate a Sigma Expression

To evaluate $\displaystyle\sum_{k=1}^{n} f(k)$:

1. Substitute $k = 1$ into $f(k)$ to get the first term.
2. Substitute $k = 2$ to get the second term.
3. Continue until $k = n$.
4. Add all the resulting terms together.

How to Write a Series in Sigma Notation

To convert a written-out series into sigma notation:

1. Find the general term $f(k)$ — the rule that produces each term when $k = 1, 2, 3, \ldots$
2. Determine the lower limit (usually $k = 1$) and the upper limit n (the number of terms).
3. Write $\displaystyle\sum_{k=1}^{n} f(k)$.

For an **arithmetic** series, the general term is linear in k: it has the form $f(k) = dk + c$ for some constants d and c.

For a **geometric** series, the general term is $f(k) = t_1 \cdot r^{k-1}$.

The Lower Limit Does Not Have to Be 1

Sometimes the sum starts at $k = 2$, $k = 3$, or another value. For example:

$$\sum_{k=3}^{7}(4k-1) = 11 + 15 + 19 + 23 + 27$$

The procedure is the same — just start substituting at the lower limit.

Pro Tip

- **The number of terms is (upper limit) − (lower limit) +1.** For $\displaystyle\sum_{k=3}^{7}$, the number of terms is $7 - 3 + 1 = 5$. Do not confuse the upper limit with the number of terms.
- **To find the general term for an arithmetic series, use $f(k) = t_1 + (k-1)d$ and simplify.** This will always produce a linear expression in k.
- **For a geometric series the general term is always $t_1 \cdot r^{k-1}$.** The exponent is $(k-1)$, not k.
- **The index variable is a dummy variable.** $\displaystyle\sum_{k=1}^{5}(2k+1)$ and $\displaystyle\sum_{j=1}^{5}(2j+1)$ are identical. The letter used does not change the value of the sum.
- **When evaluating, list every term before adding.** Students who try to add "in their head" while substituting often lose track of a term. Write each substitution on its own line.

Examples

Example 1 (Evaluating — arithmetic): Evaluate $\sum_{k=1}^{5}(2k+1)$.

Step 1: Substitute each value of k from 1 to 5 into $(2k+1)$.

$$\begin{aligned} k=1: &\quad 2(1)+1=3 \\ k=2: &\quad 2(2)+1=5 \\ k=3: &\quad 2(3)+1=7 \\ k=4: &\quad 2(4)+1=9 \\ k=5: &\quad 2(5)+1=11 \end{aligned}$$

Step 2: Add all the terms.

$$3+5+7+9+11=35$$

$$\boxed{\sum_{k=1}^{5}(2k+1)=35}$$

Example 2 (Evaluating — geometric): Evaluate $\sum_{k=1}^{4} 3\cdot 2^{k-1}$.

Step 1: Substitute each value of k from 1 to 4.

$$\begin{aligned} k=1: &\quad 3\cdot 2^0 = 3\times 1 = 3 \\ k=2: &\quad 3\cdot 2^1 = 3\times 2 = 6 \\ k=3: &\quad 3\cdot 2^2 = 3\times 4 = 12 \\ k=4: &\quad 3\cdot 2^3 = 3\times 8 = 24 \end{aligned}$$

Step 2: Add all the terms.

$$3+6+12+24=45$$

$$\boxed{\sum_{k=1}^{4} 3\cdot 2^{k-1}=45}$$

Example 3 (Non-standard lower limit): Evaluate $\sum_{k=3}^{7}(4k-1)$.

Step 1: Note the lower limit is $k=3$, not $k=1$. The number of terms is $7-3+1=5$.
Step 2: Substitute $k=3,4,5,6,7$.

$$\begin{aligned} k=3: &\quad 4(3)-1=11 \\ k=4: &\quad 4(4)-1=15 \\ k=5: &\quad 4(5)-1=19 \\ k=6: &\quad 4(6)-1=23 \\ k=7: &\quad 4(7)-1=27 \end{aligned}$$

Step 3: Add.

$$11+15+19+23+27=95$$

$$\boxed{\sum_{k=3}^{7}(4k-1)=95}$$

Example 4 (Writing in sigma notation — arithmetic): Write $5+9+13+17+21$ in sigma notation.

Step 1: Identify t_1 and d.

$$t_1=5 \qquad d=4$$

Step 2: Write the general term using $f(k)=t_1+(k-1)d$ and simplify.

$$f(k)=5+(k-1)(4)=5+4k-4=4k+1$$

Step 3: Verify. Substituting $k=1,2,3,4,5$ gives $5,9,13,17,21$. ✓
Step 4: There are 5 terms, so the upper limit is 5.

$$\boxed{\sum_{k=1}^{5}(4k+1)}$$

Example 5 (Writing in sigma notation — geometric): Write $2+6+18+54+162$ in sigma notation.

Step 1: Identify t_1 and r.

$$t_1 = 2 \qquad r = 3$$

Step 2: Write the general term.

$$f(k) = t_1 \cdot r^{k-1} = 2 \cdot 3^{k-1}$$

Step 3: There are 5 terms. Verify: $k = 1$ gives 2, $k = 5$ gives $2 \cdot 3^4 = 162$. ✓

$$\boxed{\sum_{k=1}^{5} 2 \cdot 3^{k-1}}$$

Example 6 (Evaluating k^2): Evaluate $\sum_{k=1}^{6} k^2$.

Step 1: Substitute $k = 1$ through 6 into k^2.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36$$

Step 2: Add.

$$1 + 4 + 9 + 16 + 25 + 36 = 91$$

$$\boxed{\sum_{k=1}^{6} k^2 = 91}$$

Common Mistakes

- **Confusing the upper limit with the number of terms.** If the sum runs from $k = 3$ to $k = 7$, there are $7 - 3 + 1 = 5$ terms, not 7.
- **Using r^k instead of r^{k-1} in the general term for a geometric series.** The exponent is $(k - 1)$ so that when $k = 1$ the first term equals $t_1 \cdot r^0 = t_1$.
- **Forgetting to list every term before summing.** Trying to add while substituting leads to skipped terms. Always list first, then add.
- **Misidentifying the general term.** For an arithmetic series, test your general term $f(k)$ by substituting $k = 1, 2, 3$ to confirm it produces the correct terms before writing the final sigma expression.
- **Treating the index variable as a fixed number.** The variable k changes with each term. Students sometimes substitute only $k = 1$ and multiply, which gives the wrong answer.

Worksheet

Part A — Evaluate each sigma expression.

1. $\displaystyle\sum_{k=1}^{5}(3k-1)$

2. $\displaystyle\sum_{k=1}^{4} 5\cdot 2^{k-1}$

3. $\displaystyle\sum_{k=2}^{6}(2k+3)$

4. $\displaystyle\sum_{k=1}^{5}(-1)^k\cdot k$

   (Hint: evaluate term by term and watch the alternating signs.)

Part B — Write each series in sigma notation.

5. $3 + 7 + 11 + 15 + 19 + 23$

6. $1 + 4 + 9 + 16 + 25$

7. $6 + 12 + 24 + 48 + 96$

Part C — Mixed and challenge.

8. Show, by listing terms and adding, that $\sum_{k=1}^{n}(2k - 1) = n^2$ for $n = 5$.

9. Use the arithmetic series formula to evaluate $\sum_{k=1}^{100} k$.

   (Hint: this is $1 + 2 + 3 + \cdots + 100$ with $t_1 = 1$ and $d = 1$.)

10. *(Challenge)* Evaluate $\sum_{k=1}^{10} 2 \cdot 3^{k-1}$ using the geometric series formula. Show all steps.

Answer Key

1. $\sum_{k=1}^{5}(3k-1)$

Step 1: Substitute $k=1$ through 5.

$$k=1:\ 3(1)-1=2 \qquad k=2:\ 3(2)-1=5 \qquad k=3:\ 3(3)-1=8$$
$$k=4:\ 3(4)-1=11 \qquad k=5:\ 3(5)-1=14$$

Step 2: Add.

$$2+5+8+11+14=40$$

$$\boxed{\sum_{k=1}^{5}(3k-1)=40}$$

2. $\sum_{k=1}^{4}5\cdot 2^{k-1}$

Step 1: Substitute $k=1$ through 4.

$$k=1:\ 5\cdot 2^0=5 \qquad k=2:\ 5\cdot 2^1=10 \qquad k=3:\ 5\cdot 2^2=20 \qquad k=4:\ 5\cdot 2^3=40$$

Step 2: Add.

$$5+10+20+40=75$$

$$\boxed{\sum_{k=1}^{4}5\cdot 2^{k-1}=75}$$

3. $\sum_{k=2}^{6}(2k+3)$

Step 1: Note the lower limit is $k=2$. Number of terms: $6-2+1=5$.
Step 2: Substitute $k=2$ through 6.

$$k=2:\ 2(2)+3=7 \qquad k=3:\ 2(3)+3=9 \qquad k=4:\ 2(4)+3=11$$
$$k=5:\ 2(5)+3=13 \qquad k=6:\ 2(6)+3=15$$

Step 3: Add.

$$7+9+11+13+15=55$$

$$\boxed{\sum_{k=2}^{6}(2k+3)=55}$$

4. $\sum_{k=1}^{5}(-1)^k\cdot k$

Step 1: Evaluate term by term. When k is odd, $(-1)^k=-1$. When k is even, $(-1)^k=+1$.

$$k=1:\ (-1)^1\cdot 1=-1$$
$$k=2:\ (-1)^2\cdot 2=+2$$
$$k=3:\ (-1)^3\cdot 3=-3$$
$$k=4:\ (-1)^4\cdot 4=+4$$
$$k=5:\ (-1)^5\cdot 5=-5$$

Step 2: Add, taking signs into account.

$$-1+2-3+4-5=-3$$

$$\boxed{\sum_{k=1}^{5}(-1)^k\cdot k=-3}$$

5. $3 + 7 + 11 + 15 + 19 + 23$
Step 1: Identify $t_1 = 3$ and $d = 4$.
Step 2: Find the general term.

$$f(k) = 3 + (k-1)(4) = 3 + 4k - 4 = 4k - 1$$

Step 3: Verify by substituting $k = 1$ and $k = 6$.

$$f(1) = 4(1) - 1 = 3\checkmark \qquad f(6) = 4(6) - 1 = 23\checkmark$$

Step 4: There are 6 terms, so the upper limit is 6.

$$\boxed{\sum_{k=1}^{6}(4k-1)}$$

6. $1 + 4 + 9 + 16 + 25$
Step 1: Recognise the pattern. These are perfect squares: $1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2$.
Step 2: The general term is $f(k) = k^2$ and there are 5 terms.

$$\boxed{\sum_{k=1}^{5} k^2}$$

7. $6 + 12 + 24 + 48 + 96$
Step 1: Identify $t_1 = 6$ and $r = 2$.
Step 2: The general term for a geometric series is $t_1 \cdot r^{k-1}$.

$$f(k) = 6 \cdot 2^{k-1}$$

Step 3: Verify. $k = 1$**:** $6 \cdot 2^0 = 6\ \checkmark \quad k = 5$**:** $6 \cdot 2^4 = 96\ \checkmark$
Step 4: There are 5 terms.

$$\boxed{\sum_{k=1}^{5} 6 \cdot 2^{k-1}}$$

8. Show that $\sum_{k=1}^{5}(2k-1) = 5^2 = 25$.

Step 1: Substitute $k = 1$ through 5 into $(2k-1)$.

$$\begin{aligned}
k = 1 &: \ 2(1) - 1 = 1 \\
k = 2 &: \ 2(2) - 1 = 3 \\
k = 3 &: \ 2(3) - 1 = 5 \\
k = 4 &: \ 2(4) - 1 = 7 \\
k = 5 &: \ 2(5) - 1 = 9
\end{aligned}$$

Step 2: Add.

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2 \checkmark$$

$$\boxed{\sum_{k=1}^{5}(2k-1) = 25 = 5^2}$$

This result holds for any n: the sum of the first n odd numbers always equals n^2.

9. $\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 100$

Step 1: Recognise this as an arithmetic series with $t_1 = 1$, $t_{100} = 100$, $n = 100$.

Step 2: Use the arithmetic series Formula 2.

$$S_{100} = \frac{100}{2}(t_1 + t_{100}) = 50 \times (1 + 100) = 50 \times 101$$

$$\boxed{\sum_{k=1}^{100} k = 5050}$$

10. (Challenge) $\sum_{k=1}^{10} 2 \cdot 3^{k-1}$

Step 1: Recognise this as a geometric series with $t_1 = 2$, $r = 3$, $n = 10$.

Step 2: Apply the geometric series formula.

$$S_n = \frac{t_1\left(r^n - 1\right)}{r - 1}$$

$$S_{10} = \frac{2\left(3^{10} - 1\right)}{3 - 1} = \frac{2\left(3^{10} - 1\right)}{2} = 3^{10} - 1$$

Step 3: Evaluate 3^{10}.

$$3^{10} = 59049$$

$$S_{10} = 59049 - 1$$

$$\boxed{\sum_{k=1}^{10} 2 \cdot 3^{k-1} = 59048}$$

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