Annuities: Present Value
ApexMath

Notes

The Other Side of Annuities

In the previous lesson, we asked: *if I make regular deposits, how much will I have in the future?* That was future value. Now we flip the question: *if I want to receive regular payments in the future, how much money do I need right now to make that possible?* This is the **present value** of an annuity.

Present value answers questions like:

- How large a loan can I afford if I can make payments of $X per month?
- How much must I have saved today to withdraw a fixed amount every month for a set number of years?
- What is a series of future payments worth in today's dollars?

Key Vocabulary

- **Payment** (R): The fixed amount received or paid at the end of each period. Every period, the same amount.
- **Interest Rate per Period** (i): The annual rate divided by the compounding frequency: $i = r \div n_{\text{freq}}$
- **Number of Payments** (n): The total number of payment periods: $n = n_{\text{freq}} \times t$
- **Present Value** (PV): The single lump-sum amount today that is equivalent in value to all future payments combined.

The Present Value Annuity Formula

$$PV = R \cdot \frac{1 - (1+i)^{-n}}{i}$$

The negative exponent $(1+i)^{-n}$ means $\dfrac{1}{(1+i)^n}$, which is the same growth factor from compound interest, but used to *discount* future payments back to today's value.

Solving for the Payment R

If you know the loan amount (present value) and want to find the required payment, rearrange the formula by multiplying both sides by $\dfrac{i}{1-(1+i)^{-n}}$:

$$R = PV \cdot \frac{i}{1 - (1+i)^{-n}}$$

This is the formula used to calculate monthly car payments, mortgage payments, and any other fixed loan repayment.

Future Value vs. Present Value — Side by Side

Future Value	**Present Value**
You make payments and watch them grow.	You borrow a lump sum and pay it back.
$FV = R \cdot \frac{(1+i)^n - 1}{i}$	$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$
Used for: savings plans, investments.	Used for: loans, mortgages, car payments.

Pro Tip

- The only structural difference between the FV and PV formulas is the numerator: $(1+i)^n - 1$ for future value versus $1-(1+i)^{-n}$ for present value. If you know one formula, the other is easy to remember.
- $(1+i)^{-n}$ means the same as $\frac{1}{(1+i)^n}$. Compute $(1+i)^n$ first on your calculator, then press the reciprocal key ($1/x$ or x^{-1}) to get $(1+i)^{-n}$.
- When solving for R, compute the denominator $1-(1+i)^{-n}$ first, then divide $PV \times i$ by it. Breaking it into two steps avoids bracket mistakes.
- Always check: $R \times n$ should be *greater* than PV. You always pay back more than you borrowed because of interest. The difference $R \times n - PV$ is the total interest paid over the life of the loan.
- As with all compound interest work, never round i partway through the calculation. Keep full precision until the final answer.

Examples

Example 1 (Find the Present Value): What lump sum today is equivalent to receiving \$300 per month for 4 years, if money earns 6% per year compounded monthly?

Step 1: Identify the values.

$$R = 300, \qquad r = 6\% = 0.06, \qquad n_{\text{freq}} = 12, \qquad t = 4 \text{ years}$$

Step 2: Find i and n.

$$i = \frac{0.06}{12} = 0.005 \qquad \text{and} \qquad n = 12 \times 4 = 48$$

Step 3: Compute $(1+i)^{-n}$.

$$(1.005)^{-48} \approx 0.787098$$

Step 4: Substitute into the formula.

$$PV = 300 \cdot \frac{1 - 0.787098}{0.005} = 300 \cdot \frac{0.212902}{0.005} = 300 \times 42.580$$

$$\boxed{PV \approx \$12{,}774.10}$$

Example 2 (Find the Present Value): An investment pays \$500 at the end of every quarter for 5 years. The annual interest rate is 8%, compounded quarterly. Find the present value.

Step 1: Identify the values.

$$R = 500, \qquad r = 8\% = 0.08, \qquad n_{\text{freq}} = 4, \qquad t = 5 \text{ years}$$

Step 2: Find i and n.

$$i = \frac{0.08}{4} = 0.02 \qquad \text{and} \qquad n = 4 \times 5 = 20$$

Step 3: Compute $(1.02)^{-20}$.

$$(1.02)^{-20} \approx 0.672971$$

Step 4: Substitute into the formula.

$$PV = 500 \cdot \frac{1 - 0.672971}{0.02} = 500 \cdot \frac{0.327029}{0.02} = 500 \times 16.351$$

$$\boxed{PV \approx \$8{,}175.72}$$

Example 3 (Find the Present Value): An annuity pays \$1,000 every 6 months for 10 years. The annual interest rate is 5%, compounded semi-annually. What is the present value?

Step 1: Identify the values.

$$R = 1000, \qquad r = 5\% = 0.05, \qquad n_{\text{freq}} = 2, \qquad t = 10 \text{ years}$$

Step 2: Find i and n.

$$i = \frac{0.05}{2} = 0.025 \qquad \text{and} \qquad n = 2 \times 10 = 20$$

Step 3: Compute $(1.025)^{-20}$.

$$(1.025)^{-20} \approx 0.610271$$

Step 4: Substitute into the formula.

$$PV = 1000 \cdot \frac{1 - 0.610271}{0.025} = 1000 \cdot \frac{0.389729}{0.025} = 1000 \times 15.589$$

$$\boxed{PV \approx \$15{,}589.16}$$

Example 4 (Solve for the Payment R): You take out a \$20,000 loan at 6% per year, compounded monthly, to be repaid over 5 years. What is your monthly payment?

Step 1: Identify the values. R is the unknown.

$$PV = 20000, \qquad r = 6\% = 0.06, \qquad n_{\text{freq}} = 12, \qquad t = 5 \text{ years}$$

Step 2: Find i and n.

$$i = \frac{0.06}{12} = 0.005 \qquad \text{and} \qquad n = 12 \times 5 = 60$$

Step 3: Compute $(1.005)^{-60}$.

$$(1.005)^{-60} \approx 0.741372$$

Step 4: Substitute into the formula for R.

$$R = 20000 \cdot \frac{0.005}{1 - 0.741372} = 20000 \cdot \frac{0.005}{0.258628} = \frac{100}{0.258628}$$

$$\boxed{R \approx \$386.66 \text{ per month}}$$

Example 5 (Find the Present Value): A car dealership offers monthly payments of \$800 for 3 years. If the interest rate is 7% per year, compounded monthly, what is the cash price of the car today?

Step 1: Identify the values.

$$R = 800, \qquad r = 7\% = 0.07, \qquad n_{\text{freq}} = 12, \qquad t = 3 \text{ years}$$

Step 2: Find i and n.

$$i = \frac{0.07}{12} \approx 0.005833 \qquad \text{and} \qquad n = 12 \times 3 = 36$$

Step 3: Compute $(1.005833)^{-36}$.

$$(1.005833)^{-36} \approx 0.811079$$

Step 4: Substitute into the formula.

$$PV = 800 \cdot \frac{1 - 0.811079}{0.005833} = 800 \cdot \frac{0.188921}{0.005833} = 800 \times 32.386$$

$$\boxed{PV \approx \$25{,}909.17}$$

Common Mistakes

- **Using the future value formula instead of the present value formula.** The two formulas look similar. Always check: are you starting with payments and finding a lump sum today (PV), or building up savings for the future (FV)?
- **Forgetting that $(1+i)^{-n}$ is a reciprocal.** A negative exponent does not make the number negative. $(1.005)^{-48}$ is a positive number slightly less than 1, approximately 0.787.
- **Using r instead of i in the formula.** As always, the rate per period $i = r \div n_{\text{freq}}$ must be used, not the annual rate r directly.
- **Using the number of years instead of total payments for n.** For monthly payments over 4 years, $n = 48$, not 4.
- **Expecting $R \times n$ to equal PV.** It never does — you always pay back more than you borrowed. If $R \times n = PV$, you have made an error somewhere.

Worksheet

1. Find the present value of monthly payments of \$400 for 3 years at 6% per year, compounded monthly.

2. Find the present value of quarterly payments of \$250 for 6 years at 8% per year, compounded quarterly.

3. Find the present value of semi-annual payments of \$1,500 for 8 years at 5% per year, compounded semi-annually.

4. You take out a \$15,000 loan at 9% per year, compounded monthly, to be repaid over 4 years. What is your monthly payment?

5. Find the present value of monthly payments of \$600 for 5 years at 4% per year, compounded monthly.

6. You borrow \$25,000 at 6% per year, compounded monthly, to be repaid over 10 years. What is your monthly payment?

7. Find the present value of quarterly payments of \$200 for 8 years at 10% per year, compounded quarterly.

8. (Challenge) You win a prize that pays \$500 per month for 4 years. The interest rate is 6% per year, compounded monthly. A friend offers to buy the prize from you for a lump sum of \$21,000 today. Should you accept? Show your work.

Answer Key

1. $R = \$400$, $r = 0.06$, $n_{\text{freq}} = 12$, $t = 3$ years.

Step 1: Find i and n.

$$i = \frac{0.06}{12} = 0.005 \qquad \text{and} \qquad n = 12 \times 3 = 36$$

Step 2: Compute $(1.005)^{-36}$.

$$(1.005)^{-36} \approx 0.836017$$

Step 3: Substitute into the formula.

$$PV = 400 \cdot \frac{1 - 0.836017}{0.005} = 400 \cdot \frac{0.163983}{0.005} = 400 \times 32.871$$

$$\boxed{PV \approx \$13{,}148.41}$$

2. $R = \$250$, $r = 0.08$, $n_{\text{freq}} = 4$, $t = 6$ years.

Step 1: Find i and n.

$$i = \frac{0.08}{4} = 0.02 \qquad \text{and} \qquad n = 4 \times 6 = 24$$

Step 2: Compute $(1.02)^{-24}$.

$$(1.02)^{-24} \approx 0.621722$$

Step 3: Substitute into the formula.

$$PV = 250 \cdot \frac{1 - 0.621722}{0.02} = 250 \cdot \frac{0.378278}{0.02} = 250 \times 18.914$$

$$\boxed{PV \approx \$4{,}728.48}$$

3. $R = \$1{,}500$, $r = 0.05$, $n_{\text{freq}} = 2$, $t = 8$ years.

Step 1: Find i and n.

$$i = \frac{0.05}{2} = 0.025 \qquad \text{and} \qquad n = 2 \times 8 = 16$$

Step 2: Compute $(1.025)^{-16}$.

$$(1.025)^{-16} \approx 0.676839$$

Step 3: Substitute into the formula.

$$PV = 1500 \cdot \frac{1 - 0.676839}{0.025} = 1500 \cdot \frac{0.323161}{0.025} = 1500 \times 12.926$$

$$\boxed{PV \approx \$19{,}582.50}$$

4. $PV = \$15{,}000$, $r = 0.09$, $n_{\text{freq}} = 12$, $t = 4$ years. Find R.

Step 1: Find i and n.

$$i = \frac{0.09}{12} = 0.0075 \qquad \text{and} \qquad n = 12 \times 4 = 48$$

Step 2: Compute $(1.0075)^{-48}$.

$$(1.0075)^{-48} \approx 0.698614$$

Step 3: Substitute into the formula for R.

$$R = 15000 \cdot \frac{0.0075}{1 - 0.698614} = 15000 \cdot \frac{0.0075}{0.301386} = \frac{112.50}{0.301386}$$

$$\boxed{R \approx \$373.28 \text{ per month}}$$

5. $R = \$600$, $r = 0.04$, $n_{\text{freq}} = 12$, $t = 5$ years.

Step 1: Find i and n.

$$i = \frac{0.04}{12} \approx 0.003333 \qquad \text{and} \qquad n = 12 \times 5 = 60$$

Step 2: Compute $(1.003333)^{-60}$.

$$(1.003333)^{-60} \approx 0.818731$$

Step 3: Substitute into the formula.

$$PV = 600 \cdot \frac{1 - 0.818731}{0.003333} = 600 \cdot \frac{0.181269}{0.003333} = 600 \times 54.382$$

$$\boxed{PV \approx \$32{,}579.44}$$

6. $PV = \$25{,}000$, $r = 0.06$, $n_{\text{freq}} = 12$, $t = 10$ years. Find R.

Step 1: Find i and n.

$$i = \frac{0.06}{12} = 0.005 \qquad \text{and} \qquad n = 12 \times 10 = 120$$

Step 2: Compute $(1.005)^{-120}$.

$$(1.005)^{-120} \approx 0.550450$$

Step 3: Substitute into the formula for R.

$$R = 25000 \cdot \frac{0.005}{1 - 0.550450} = 25000 \cdot \frac{0.005}{0.449550} = \frac{125}{0.449550}$$

$$\boxed{R \approx \$277.55 \text{ per month}}$$

7. $R = \$200$, $r = 0.10$, $n_{\text{freq}} = 4$, $t = 8$ years.

Step 1: Find i and n.

$$i = \frac{0.10}{4} = 0.025 \qquad \text{and} \qquad n = 4 \times 8 = 32$$

Step 2: Compute $(1.025)^{-32}$.

$$(1.025)^{-32} \approx 0.453771$$

Step 3: Substitute into the formula.

$$PV = 200 \cdot \frac{1 - 0.453771}{0.025} = 200 \cdot \frac{0.546229}{0.025} = 200 \times 21.849$$

$$\boxed{PV \approx \$4{,}369.84}$$

8. (Challenge) $R = \$500$, $r = 0.06$, $n_{\text{freq}} = 12$, $t = 4$ years.

Step 1: Find i and n.

$$i = \frac{0.06}{12} = 0.005 \qquad \text{and} \qquad n = 12 \times 4 = 48$$

Step 2: Compute $(1.005)^{-48}$.

$$(1.005)^{-48} \approx 0.787098$$

Step 3: Find the present value of the prize.

$$PV = 500 \cdot \frac{1 - 0.787098}{0.005} = 500 \cdot \frac{0.212902}{0.005} = 500 \times 42.580$$

$$PV \approx \$21{,}290.16$$

Step 4: Compare to the lump sum offer of \$21,000.

$$\$21{,}290.16 > \$21{,}000$$

$$\boxed{\text{Do not accept. The prize is worth \$21,290.16 today, which is \$290.16 more than the offer.}}$$

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