Applications of Quadratic Functions
ApexMath

Notes

The Big Idea

Quadratic functions appear constantly in the real world. In this lesson you will apply everything you have learned — setting up equations, finding vertices, solving by factoring, and using the quadratic formula — to solve three families of word problems: **area problems**, **projectile motion problems**, and **revenue problems**.

Every word problem follows the same general strategy:

1. **Define your variable.** Choose one unknown, call it x, and write a clear sentence stating what x represents, including its units.
2. **Build the equation.** Express all other quantities in terms of x, then write the equation or function that models the situation.
3. **Solve.** Use the appropriate technique: factoring, the quadratic formula, or finding the vertex.
4. **Interpret.** Check whether your answer makes sense in context. Reject any solutions that are physically impossible (e.g. negative lengths or times).

Area Problems

Area problems typically give you a constraint (such as a fixed perimeter or a total area) and ask you to find dimensions or a maximum area. You set up one variable, express the second dimension in terms of it, multiply to get an area function, then find the vertex (for maximum area) or solve a quadratic equation (for a specific area).

Projectile Motion

When an object is launched upward, its height h (in metres or feet) at time t (in seconds) is modelled by:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0$$

where g is the gravitational constant ($g \approx 9.8$ m/s^2 or $g = 32$ ft/s^2), v_0 is the initial velocity, and h_0 is the initial height. In most textbook problems you will see either:

$$h(t) = -4.9t^2 + v_0 t + h_0 \quad \text{(metric)} \qquad \text{or} \qquad h(t) = -16t^2 + v_0 t + h_0 \quad \text{(imperial)}$$

- **Maximum height:** Find the vertex. The t-coordinate of the vertex is when the maximum occurs; the h-coordinate is the maximum height.
- **When does the object hit the ground?** Set $h(t) = 0$ and solve. Reject the negative time solution.

- **Height at a specific time:** Substitute that value of t directly.

Revenue Problems
Revenue is the amount of money earned from sales:

$$\text{Revenue} = \text{Price} \times \text{Quantity sold}$$

When the price increases, fewer units are sold, and vice versa. This relationship is usually given as a linear demand equation. Substituting it into the revenue formula gives a quadratic function. You then find its vertex to maximise revenue.

Pro Tip

- Always define your variable before writing any equation. A sentence like "Let x = the width of the garden in metres" keeps the algebra organised and makes your solution readable.
- For area problems, draw and label a diagram. This makes it much easier to see how the dimensions relate to each other and to the constraint.
- For projectile problems, write down what each part of the formula represents before substituting. Confusing v_0 and h_0 is a very common error.
- After solving a quadratic equation in context, always check both solutions against the real-world constraints. Negative time, negative length, and negative quantity are all physically impossible and must be rejected.
- For maximum or minimum problems, remember: the vertex gives the optimal value. The x-coordinate of the vertex tells you *when* or *at what input* the optimum occurs, and the y-coordinate tells you *what* the optimum value is.

Examples

Example 1 (Area — fixed perimeter): A farmer has 80 metres of fencing to enclose a rectangular garden. What dimensions maximise the area, and what is the maximum area?

Step 1: Define the variable.
Let x = the width of the garden in metres.
Step 2: Build the equation.
The perimeter of the rectangle is 80 m:

$$2x + 2l = 80 \qquad \Longrightarrow \qquad l = 40 - x$$

where l is the length. The area is:

$$A(x) = x \cdot l = x(40 - x) = 40x - x^2$$

Step 3: Find the maximum. Rewrite in vertex form by completing the square.

$$A(x) = -x^2 + 40x = -(x^2 - 40x)$$

$$= -(x^2 - 40x + 400 - 400) = -(x - 20)^2 + 400$$

The vertex is $(20, 400)$.
Step 4: Interpret.
The maximum area is 400 m^2, achieved when $x = 20$ m. The length is $l = 40 - 20 = 20$ m.
The garden is a square with side 20 m.

$$\boxed{\text{Dimensions: } 20 \text{ m} \times 20 \text{ m}, \quad \text{Maximum area: } 400 \text{ m}^2}$$

Example 2 (Area — specific area given): A rectangular garden has a length that is 3 metres more than its width. The area of the garden is 40 m^2. Find the dimensions.

Step 1: Let $x =$ the width in metres.
Step 2: The length is $x + 3$. Set up the area equation.

$$x(x + 3) = 40$$

$$x^2 + 3x - 40 = 0$$

Step 3: Factor.

$$(x + 8)(x - 5) = 0$$

$$x = -8 \qquad \text{or} \qquad x = 5$$

Step 4: Reject $x = -8$ since width cannot be negative.
Width = 5 m, Length = $5 + 3 = 8$ m.
Verify: $5 \times 8 = 40$ m^2. ✓

$$\boxed{\text{Width: } 5 \text{ m}, \quad \text{Length: } 8 \text{ m}}$$

Example 3 (Projectile — maximum height): A ball is thrown upward from the ground with an initial velocity of 24 m/s. Its height in metres after t seconds is:

$$h(t) = -4.9t^2 + 24t$$

Find the maximum height and the time at which it occurs.

Step 1: Identify $a = -4.9$, $b = 24$, $c = 0$.
Step 2: Find the t-coordinate of the vertex using $t = -\dfrac{b}{2a}$.

$$t = -\frac{24}{2(-4.9)} = -\frac{24}{-9.8} = \frac{24}{9.8} \approx 2.449 \text{ s}$$

Step 3: Substitute back to find the maximum height.

$$h(2.449) = -4.9(2.449)^2 + 24(2.449)$$

$$= -4.9(5.998) + 58.776$$

$$\approx -29.390 + 58.776 \approx 29.39 \text{ m}$$

Step 4: Interpret.
The ball reaches a maximum height of approximately 29.4 m at about 2.45 seconds.

$$\boxed{\text{Maximum height} \approx 29.4 \text{ m at } t \approx 2.45 \text{ s}}$$

Example 4 (Projectile — when does it land): A ball is launched from a platform 5 m above the ground with an initial upward velocity of 20 m/s. Its height is:

$$h(t) = -4.9t^2 + 20t + 5$$

When does the ball hit the ground?

Step 1: Set $h(t) = 0$.

$$-4.9t^2 + 20t + 5 = 0$$

Step 2: Multiply both sides by -1 to make the leading coefficient positive.

$$4.9t^2 - 20t - 5 = 0$$

Step 3: Apply the quadratic formula with $a = 4.9$, $b = -20$, $c = -5$.

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4.9)(-5)}}{2(4.9)} = \frac{20 \pm \sqrt{400 + 98}}{9.8} = \frac{20 \pm \sqrt{498}}{9.8}$$

$$\sqrt{498} \approx 22.316$$

$$t = \frac{20 + 22.316}{9.8} \approx \frac{42.316}{9.8} \approx 4.32 \text{ s} \qquad \text{or} \qquad t = \frac{20 - 22.316}{9.8} \approx \frac{-2.316}{9.8} \approx -0.24 \text{ s}$$

Step 4: Reject $t \approx -0.24$ since time cannot be negative.

$$\boxed{\text{The ball hits the ground at } t \approx 4.32 \text{ s}}$$

Example 5 (Revenue): A shop currently sells 200 items per week at \$50 each. Market research shows that for every \$5 increase in price, the shop sells 10 fewer items per week. What price maximises weekly revenue?

Step 1: Let $x =$ the number of \$5 price increases.
Step 2: Build expressions for price and quantity.

$$\text{Price} = 50 + 5x$$

$$\text{Quantity} = 200 - 10x$$

Step 3: Write the revenue function.

$$R(x) = (50 + 5x)(200 - 10x)$$

Expand:

$$R(x) = 10000 - 500x + 1000x - 50x^2$$
$$R(x) = -50x^2 + 500x + 10000$$

Step 4: Find the vertex. Use $x = -\dfrac{b}{2a}$.

$$x = -\frac{500}{2(-50)} = -\frac{500}{-100} = 5$$

Step 5: Interpret.
The optimal number of \$5 price increases is 5.
Price $= 50 + 5(5) =$ \$75.
Quantity $= 200 - 10(5) = 150$ items.
Maximum revenue $= 75 \times 150 =$ \$11,250 per week.

$$\boxed{\text{Optimal price: \$75,} \quad \text{Maximum revenue: \$11,250/week}}$$

Common Mistakes

- **Not defining the variable clearly:** Writing down equations without stating what x represents leads to confusion about which solution is valid and what the answer means in context.
- **Forgetting to reject invalid solutions:** After solving a quadratic in context, always check both solutions. Negative dimensions, negative time, and negative quantities are not physically meaningful.
- **Confusing the vertex coordinates:** The x-coordinate of the vertex is the input (time, width, number of price increases) that produces the maximum or minimum. The y-coordinate is the actual maximum or minimum value (height, area, revenue). Students often report the x-coordinate as the answer when the question asks for the maximum value.
- **Setting up the wrong constraint in area problems:** Make sure you are using the correct perimeter formula for the shape described. A garden fenced on only three sides (against a wall) has a different constraint than one fenced on all four sides.
- **Using the wrong formula for projectile motion:** Always check whether the problem uses metric or imperial units and use $-4.9t^2$ or $-16t^2$ accordingly. Using -9.8 instead of -4.9 is a common error (the $\frac{1}{2}$ factor is already built into 4.9).

Worksheet

Area Problems

1. A farmer has 60 m of fencing to enclose a rectangular paddock. What dimensions give the maximum area, and what is that maximum area?

2. A rectangle has a length that is 4 cm more than its width. Its area is 96 cm^2. Find the dimensions.

3. A rectangular garden is to be built against a wall, so fencing is only needed on three sides. The total fencing available is 48 m. Find the dimensions that maximise the area.

Projectile Motion Problems

4. A ball is thrown upward from the ground with an initial velocity of 30 m/s. Its height is $h(t) = -4.9t^2 + 30t$. Find the maximum height and the time it occurs.

5. A rocket is launched from the ground with an initial velocity of 40 m/s. Its height is $h(t) = -4.9t^2 + 40t$. At what time(s) is the rocket at a height of 60 m?

6. A stone is dropped from a cliff 80 m high with zero initial velocity. Its height is $h(t) = -4.9t^2 + 80$. How long does it take to hit the ground? Round to two decimal places.

7. A ball is thrown upward from a rooftop 15 m above the ground with an initial velocity of 10 m/s. Its height is $h(t) = -4.9t^2 + 10t + 15$. When does it hit the ground? Round to two decimal places.

Revenue Problems

8. A shop sells 300 items per week at \$20 each. For every \$2 increase in price, 15 fewer items are sold per week. Find the price that maximises weekly revenue.

9. A cinema currently sells 500 tickets per night at \$12 each. For every \$1 increase in ticket price, 20 fewer tickets are sold. Find the ticket price that maximises nightly revenue and state that maximum revenue.

10. (Challenge) A company sells x units of a product at a price of $(80 - 0.5x)$ dollars per unit. The cost to produce x units is $(10x + 400)$ dollars. The profit is Revenue $-$ Cost. Find the number of units that maximises profit and state the maximum profit.

Answer Key

1. Fencing = 60 m, four-sided rectangle.
Let x = width in metres. Then length $= \dfrac{60 - 2x}{2} = 30 - x$.
Area:

$$A(x) = x(30 - x) = -x^2 + 30x$$

Vertex: $x = -\dfrac{30}{2(-1)} = 15$.
Length $= 30 - 15 = 15$ m.
Maximum area $= 15 \times 15 = 225$ m^2.

$$\boxed{\text{Dimensions: 15 m} \times \text{15 m}, \quad \text{Maximum area: 225 m}^2}$$

2. Length is 4 cm more than width, area = 96 cm^2.
Let x = width. Length $= x + 4$.

$$x(x + 4) = 96$$

$$x^2 + 4x - 96 = 0$$

Use the quadratic formula with $a = 1$, $b = 4$, $c = -96$:

$$\text{Discriminant: } 4^2 - 4(1)(-96) = 16 + 384 = 400$$

$$x = \frac{-4 \pm \sqrt{400}}{2} = \frac{-4 \pm 20}{2}$$

$$x = \frac{-4 + 20}{2} = 8 \qquad \text{or} \qquad x = \frac{-4 - 20}{2} = -12$$

Reject $x = -12$. Width = 8 cm, Length = 12 cm.
Verify: $8 \times 12 = 96$ cm^2. ✓

$$\boxed{\text{Width: 8 cm,} \quad \text{Length: 12 cm}}$$

3. Three-sided fence, total fencing = 48 m.

Let x = the dimension parallel to the wall (the side with no fence). The two sides perpendicular to the wall each have length w.

The fencing constraint is:

$$x + 2w = 48 \qquad \Longrightarrow \qquad x = 48 - 2w$$

Area:

$$A(w) = w \cdot x = w(48 - 2w) = -2w^2 + 48w$$

Vertex: $w = -\dfrac{48}{2(-2)} = 12$.
Then $x = 48 - 2(12) = 24$ m.
Maximum area = $12 \times 24 = 288$ m^2.

$$\boxed{\text{Dimensions: 24 m (along wall)} \times \text{12 m,} \quad \text{Maximum area: 288 m}^2}$$

4. $h(t) = -4.9t^2 + 30t$.
Time of maximum: $t = -\dfrac{30}{2(-4.9)} = \dfrac{30}{9.8} \approx 3.061$ s.
Maximum height:

$$h(3.061) = -4.9(3.061)^2 + 30(3.061) \approx -4.9(9.370) + 91.837 \approx -45.913 + 91.837 \approx 45.92 \text{ m}$$

$$\boxed{\text{Maximum height} \approx 45.9 \text{ m at } t \approx 3.06 \text{ s}}$$

5. $h(t) = -4.9t^2 + 40t = 60$.
Set up:

$$-4.9t^2 + 40t - 60 = 0$$

Multiply by -1:

$$4.9t^2 - 40t + 60 = 0$$

Quadratic formula with $a = 4.9$, $b = -40$, $c = 60$:

$$\text{Discriminant: } (-40)^2 - 4(4.9)(60) = 1600 - 1176 = 424$$

$$t = \frac{40 \pm \sqrt{424}}{9.8}$$

$$\sqrt{424} \approx 20.591$$

$$t = \frac{40 + 20.591}{9.8} \approx \frac{60.591}{9.8} \approx 6.18 \text{ s} \quad \text{or} \quad t = \frac{40 - 20.591}{9.8} \approx \frac{19.409}{9.8} \approx 1.98 \text{ s}$$

Both times are positive. The rocket passes 60 m on the way up at $t \approx 1.98$ s and on the way down at $t \approx 6.18$ s.

$$\boxed{t \approx 1.98 \text{ s (ascending) and } t \approx 6.18 \text{ s (descending)}}$$

6. $h(t) = -4.9t^2 + 80 = 0$.

$$4.9t^2 = 80$$

$$t^2 = \frac{80}{4.9} \approx 16.327$$

$$t = \sqrt{16.327} \approx 4.04 \text{ s}$$

Reject the negative root.

$$\boxed{t \approx 4.04 \text{ s}}$$

7. $h(t) = -4.9t^2 + 10t + 15 = 0$.
Multiply by -1:

$$4.9t^2 - 10t - 15 = 0$$

Quadratic formula with $a = 4.9$, $b = -10$, $c = -15$:

$$\text{Discriminant: } (-10)^2 - 4(4.9)(-15) = 100 + 294 = 394$$

$$t = \frac{10 \pm \sqrt{394}}{9.8}$$

$$\sqrt{394} \approx 19.849$$

$$t = \frac{10 + 19.849}{9.8} \approx \frac{29.849}{9.8} \approx 3.05 \text{ s} \quad \text{or} \quad t = \frac{10 - 19.849}{9.8} \approx \frac{-9.849}{9.8} \approx -1.00 \text{ s}$$

Reject the negative time.

$$\boxed{t \approx 3.05 \text{ s}}$$

8. Price $= 20 + 2x$, Quantity $= 300 - 15x$.

$$R(x) = (20 + 2x)(300 - 15x) = 6000 - 300x + 600x - 30x^2 = -30x^2 + 300x + 6000$$

Vertex: $x = -\dfrac{300}{2(-30)} = -\dfrac{300}{-60} = 5.$
Price $= 20 + 2(5) = \$30$.
Quantity $= 300 - 15(5) = 225$.
Maximum revenue $= 30 \times 225 = \$6{,}750$.

$$\boxed{\text{Optimal price: \$30,} \quad \text{Maximum revenue: \$6,750/week}}$$

9. Price $= 12 + x$, Quantity $= 500 - 20x$.

$$R(x) = (12 + x)(500 - 20x) = 6000 - 240x + 500x - 20x^2 = -20x^2 + 260x + 6000$$

Vertex: $x = -\dfrac{260}{2(-20)} = -\dfrac{260}{-40} = 6.5.$
Price $= 12 + 6.5 = \$18.50$.
Quantity $= 500 - 20(6.5) = 500 - 130 = 370$ tickets.
Maximum revenue $= 18.50 \times 370 = \$6{,}845$.

$$\boxed{\text{Optimal price: \$18.50,} \quad \text{Maximum revenue: \$6,845/night}}$$

10. (Challenge)
Revenue = Price × Quantity $= (80 - 0.5x) \cdot x = 80x - 0.5x^2$.
Cost $= 10x + 400$.
Profit:

$$P(x) = \text{Revenue} - \text{Cost} = (80x - 0.5x^2) - (10x + 400)$$

$$P(x) = -0.5x^2 + 70x - 400$$

Vertex: $x = -\dfrac{70}{2(-0.5)} = -\dfrac{70}{-1} = 70$ units.
Maximum profit:

$$P(70) = -0.5(70)^2 + 70(70) - 400 = -0.5(4900) + 4900 - 400 = -2450 + 4900 - 400 = 2050$$

$$\boxed{\text{70 units maximise profit; maximum profit = \$2,050}}$$

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