Completing the Square
ApexMath

Notes

What Is Completing the Square?

Completing the square is an algebraic technique that rewrites a quadratic expression into a perfect square trinomial plus a constant. The result always takes the form:

$$a(x-h)^2+k$$

You have already seen this idea used inside a graphing lesson. This lesson isolates the skill so you can practice it cleanly and deeply, because it appears in many areas of mathematics beyond graphing — including solving quadratic equations, deriving the quadratic formula, and working with conic sections.

What Is a Perfect Square Trinomial?

A perfect square trinomial is a three-term expression that factors into a binomial squared. The pattern is:

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2$$

For example:

$$x^2+6x+9=(x+3)^2 \qquad \text{because } \left(\frac{6}{2}\right)^2=9$$

The key number — the one you add to complete the square — is always $\left(\frac{b}{2}\right)^2$, where b is the coefficient of x.

The Full Process When $a=1$

Starting from x^2+bx+c:

1. Move the constant c to the other side (or set it aside).
2. Find the completing value: $\left(\frac{b}{2}\right)^2$.
3. Add and subtract the completing value to keep the expression equivalent.
4. Rewrite the three-term group as a squared binomial.
5. Combine the leftover constants.

The Full Process When $a\neq 1$

When the leading coefficient is not 1, there is one extra step at the beginning.

1. Factor a out of the first two terms only. Leave the constant alone.

2. Inside the parentheses, the coefficient of x is now $\frac{b}{a}$. Find the completing value: $\left(\frac{b}{2a}\right)^2$.

3. Add the completing value inside the parentheses. Because of the factor of a outside, this actually adds $a \cdot \left(\frac{b}{2a}\right)^2$ to the expression. Subtract this same amount outside the parentheses to compensate.

4. Rewrite the trinomial inside the parentheses as a squared binomial.

5. Combine the leftover constants.

The Compensation Rule — The Most Important Step
This rule is worth memorising on its own:

Whatever you add inside the parentheses, you must subtract $a \times$ (that amount) outside.

If $a = 1$, you add and subtract the same number. If $a = 3$ and you add 4 inside the parentheses, you must subtract $3 \times 4 = 12$ outside. If $a = -2$ and you add 9 inside, you must subtract $-2 \times 9 = -18$ outside, which is the same as adding 18.

Pro Tip

- Before you do anything, check whether the coefficient of x^2 is 1. If it is, skip the factoring step. If it is not, your very first move is to factor a out of the first two terms only.
- Write out the compensation arithmetic explicitly every time. Do not try to do it in your head. Writing "I am adding $a \cdot (\ldots)$ so I subtract $a \cdot (\ldots)$ outside" prevents the most common error in this entire lesson.
- The completing value $\left(\frac{b}{2}\right)^2$ is always non-negative because you are squaring. If you ever get a negative completing value, you have made an arithmetic error.
- After finishing, always verify your answer by expanding the vertex form back out. If you recover the original expression, you are correct.
- When the coefficient of x is a fraction, the arithmetic gets messy but the process is exactly the same. Take your time and simplify the completing value carefully before adding it.

Examples

Example 1 ($a = 1$, positive b): Complete the square for $x^2 + 10x + 3$.

Step 1: $a = 1$, so no factoring is needed.
Step 2: The coefficient of x is 10. Find the completing value.

$$\left(\frac{10}{2}\right)^2 = 5^2 = 25$$

Step 3: Add and subtract 25.

$$x^2 + 10x + 25 - 25 + 3$$

Step 4: Rewrite the perfect square trinomial.

$$(x + 5)^2 - 25 + 3$$

Step 5: Combine the constants.

$$(x + 5)^2 - 22$$

$$\boxed{(x + 5)^2 - 22}$$

Example 2 ($a = 1$, negative b): Complete the square for $x^2 - 8x + 5$.

Step 1: $a = 1$, no factoring needed.
Step 2: The coefficient of x is -8. Find the completing value.

$$\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$$

Step 3: Add and subtract 16.

$$x^2 - 8x + 16 - 16 + 5$$

Step 4: Rewrite. Notice that $x^2 - 8x + 16 = (x - 4)^2$.

$$(x - 4)^2 - 16 + 5$$

Step 5: Combine.

$$(x - 4)^2 - 11$$

$$\boxed{(x - 4)^2 - 11}$$

Example 3 ($a = 1$, negative constant): Complete the square for $x^2 + 4x - 9$.

Step 1: $a = 1$, no factoring needed.

Step 2: The coefficient of x is 4.

$$\left(\frac{4}{2}\right)^2 = 4$$

Step 3: Add and subtract 4.

$$x^2 + 4x + 4 - 4 - 9$$

Step 4: Rewrite.

$$(x+2)^2 - 4 - 9$$

Step 5: Combine.

$$(x+2)^2 - 13$$

$$\boxed{(x+2)^2 - 13}$$

Example 4 ($a \neq 1$, positive a): Complete the square for $2x^2 + 12x + 7$.

Step 1: $a = 2 \neq 1$. Factor 2 out of the first two terms only.

$$2(x^2 + 6x) + 7$$

Step 2: The coefficient of x inside the parentheses is 6.

$$\left(\frac{6}{2}\right)^2 = 9$$

Step 3: Add 9 inside the parentheses. Because the factor outside is 2, we have added $2 \times 9 = 18$ to the expression. Subtract 18 outside to compensate.

$$2(x^2 + 6x + 9) - 18 + 7$$

Step 4: Rewrite the trinomial inside.

$$2(x+3)^2 - 18 + 7$$

Step 5: Combine the constants.

$$2(x+3)^2 - 11$$

$$\boxed{2(x+3)^2 - 11}$$

Example 5 ($a \neq 1$, negative a): Complete the square for $-3x^2 + 6x + 2$.

Step 1: $a = -3 \neq 1$. Factor -3 out of the first two terms only.

$$-3(x^2 - 2x) + 2$$

Step 2: The coefficient of x inside is -2.

$$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$$

Step 3: Add 1 inside the parentheses. The factor outside is -3, so we have added $(-3) \times 1 = -3$ to the expression. Subtract -3 (i.e., add 3) outside to compensate.

$$-3(x^2 - 2x + 1) + 2 + 3$$

Step 4: Rewrite the trinomial inside.

$$-3(x-1)^2 + 2 + 3$$

Step 5: Combine.

$$-3(x-1)^2 + 5$$

$$\boxed{-3(x-1)^2 + 5}$$

Example 6 ($a \neq 1$, fractional coefficient): Complete the square for $\frac{1}{2}x^2 - 3x + 4$.

Step 1: $a = \frac{1}{2} \neq 1$. Factor $\frac{1}{2}$ out of the first two terms only.

$$\frac{1}{2}(x^2 - 6x) + 4$$

Step 2: The coefficient of x inside is -6.

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

Step 3: Add 9 inside. The factor outside is $\frac{1}{2}$, so we have added $\frac{1}{2} \times 9 = \frac{9}{2}$ to the expression. Subtract $\frac{9}{2}$ outside.

$$\frac{1}{2}(x^2 - 6x + 9) + 4 - \frac{9}{2}$$

Step 4: Rewrite.

$$\frac{1}{2}(x-3)^2 + 4 - \frac{9}{2}$$

Step 5: Combine $4 - \frac{9}{2} = \frac{8}{2} - \frac{9}{2} = -\frac{1}{2}$.

$$\frac{1}{2}(x-3)^2 - \frac{1}{2}$$

$$\boxed{\frac{1}{2}(x-3)^2 - \frac{1}{2}}$$

Common Mistakes

- **Forgetting the compensation step:** When you add the completing value inside the parentheses, the expression is no longer equal to the original. You must immediately subtract a times that value outside. Skipping this step changes the entire expression and gives a wrong answer.
- **Subtracting just the completing value instead of a times it:** If $a = 4$ and the completing value is 9, you must subtract $4 \times 9 = 36$, not 9. This is the single most common error when $a \neq 1$.
- **Factoring a out of all three terms including c:** You only factor a out of the x^2 and x terms. The constant c must stay outside the parentheses untouched. If you include it, the completing value will be wrong.
- **Forgetting to halve before squaring:** The completing value is $\left(\frac{b}{2}\right)^2$. A very common error is to compute $\left(\frac{b}{1}\right)^2 = b^2$ by forgetting to divide by 2. Always halve the coefficient of x first, then square.
- **Sign errors on b:** When the coefficient of x is negative, like -8, dividing by 2 gives -4, and squaring gives $+16$. The completing value is always positive. If you get a negative completing value, recheck your arithmetic.
- **Not verifying the answer:** Always expand your completed square back to standard form and confirm it matches the original. This takes thirty seconds and catches most errors immediately.

Worksheet

Complete the square for each expression. Write your final answer in the form $a(x-h)^2+k$.

1. $x^2 + 6x + 2$

2. $x^2 - 10x + 3$

3. $x^2 + 3x - 1$

4. $x^2 - 5x + 4$

5. $x^2 + x - 7$

6. $2x^2 + 8x + 1$

7. $3x^2 - 18x + 2$

8. $-x^2 + 4x + 6$

9. $-2x^2 - 12x + 5$

10. (Challenge) $4x^2 + 12x - 1$

11. (Challenge) $\frac{1}{3}x^2 - 2x + 5$

12. (Challenge) $-\frac{3}{2}x^2 + 6x - 1$

Answer Key

1. $x^2 + 6x + 2$

   Step 1: $a = 1$, no factoring needed.

   Step 2: Completing value: $\left(\frac{6}{2}\right)^2 = 9$.

   Step 3: Add and subtract 9.

$$x^2 + 6x + 9 - 9 + 2$$

Step 4: Rewrite.

$$(x+3)^2 - 9 + 2$$

Step 5: Combine.

$$(x+3)^2 - 7$$

$$\boxed{(x+3)^2 - 7}$$

2. $x^2 - 10x + 3$

Step 1: $a = 1$, no factoring needed.

Step 2: Completing value: $\left(\frac{-10}{2}\right)^2 = (-5)^2 = 25$.

Step 3:

$$x^2 - 10x + 25 - 25 + 3$$

Step 4:

$$(x-5)^2 - 25 + 3$$

Step 5:

$$(x-5)^2 - 22$$

$$\boxed{(x-5)^2 - 22}$$

3. $x^2 + 3x - 1$

Step 1: $a = 1$, no factoring needed.

Step 2: Completing value: $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Step 3: Add and subtract $\frac{9}{4}$.

$$x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 1$$

Step 4:

$$\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 1$$

Step 5: Combine $-\frac{9}{4} - 1 = -\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}$.

$$\left(x + \frac{3}{2}\right)^2 - \frac{13}{4}$$

$$\boxed{\left(x + \frac{3}{2}\right)^2 - \frac{13}{4}}$$

4. $x^2 - 5x + 4$

Step 1: $a = 1$, no factoring needed.

Step 2: Completing value: $\left(\frac{-5}{2}\right)^2 = \frac{25}{4}$.

Step 3:

$$x^2 - 5x + \frac{25}{4} - \frac{25}{4} + 4$$

Step 4:

$$\left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{16}{4}$$

Step 5: Combine $-\frac{25}{4} + \frac{16}{4} = -\frac{9}{4}$.

$$\left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$$

$$\boxed{\left(x - \frac{5}{2}\right)^2 - \frac{9}{4}}$$

5. $x^2 + x - 7$

Step 1: $a = 1$, no factoring needed.

Step 2: Completing value: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Step 3:

$$x^2 + x + \frac{1}{4} - \frac{1}{4} - 7$$

Step 4:

$$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - 7$$

Step 5: Combine $-\frac{1}{4} - 7 = -\frac{1}{4} - \frac{28}{4} = -\frac{29}{4}$.

$$\left(x + \frac{1}{2}\right)^2 - \frac{29}{4}$$

$$\boxed{\left(x + \frac{1}{2}\right)^2 - \frac{29}{4}}$$

6. $2x^2 + 8x + 1$

Step 1: $a = 2$. Factor 2 out of the first two terms.

$$2(x^2 + 4x) + 1$$

Step 2: Coefficient of x inside is 4. Completing value: $\left(\frac{4}{2}\right)^2 = 4$.
Step 3: Add 4 inside. Compensation: $2 \times 4 = 8$. Subtract 8 outside.

$$2(x^2 + 4x + 4) - 8 + 1$$

Step 4: Rewrite.

$$2(x + 2)^2 - 8 + 1$$

Step 5: Combine.

$$2(x + 2)^2 - 7$$

$$\boxed{2(x + 2)^2 - 7}$$

7. $3x^2 - 18x + 2$
Step 1: $a = 3$. Factor 3 out of the first two terms.

$$3(x^2 - 6x) + 2$$

Step 2: Coefficient of x inside is -6. Completing value: $\left(\frac{-6}{2}\right)^2 = 9$.
Step 3: Add 9 inside. Compensation: $3 \times 9 = 27$. Subtract 27 outside.

$$3(x^2 - 6x + 9) - 27 + 2$$

Step 4:

$$3(x - 3)^2 - 27 + 2$$

Step 5:

$$3(x - 3)^2 - 25$$

$$\boxed{3(x - 3)^2 - 25}$$

8. $-x^2 + 4x + 6$
Step 1: $a = -1$. Factor -1 out of the first two terms.

$$-(x^2 - 4x) + 6$$

Step 2: Coefficient of x inside is -4. Completing value: $\left(\frac{-4}{2}\right)^2 = 4$.
Step 3: Add 4 inside. Compensation: $(-1) \times 4 = -4$. Subtract -4 (add 4) outside.

$$-(x^2 - 4x + 4) + 6 + 4$$

Step 4:

$$-(x - 2)^2 + 6 + 4$$

Step 5:

$$-(x-2)^2+10$$

$$\boxed{-(x-2)^2+10}$$

9. $-2x^2-12x+5$
Step 1: $a=-2$. Factor -2 out of the first two terms.

$$-2(x^2+6x)+5$$

Step 2: Coefficient of x inside is 6. Completing value: $\left(\frac{6}{2}\right)^2=9$.
Step 3: Add 9 inside. Compensation: $(-2)\times 9=-18$. Subtract -18 (add 18) outside.

$$-2(x^2+6x+9)+5+18$$

Step 4:

$$-2(x+3)^2+5+18$$

Step 5:

$$-2(x+3)^2+23$$

$$\boxed{-2(x+3)^2+23}$$

10. (Challenge) $4x^2+12x-1$
Step 1: $a=4$. Factor 4 out of the first two terms.

$$4(x^2+3x)-1$$

Step 2: Coefficient of x inside is 3. Completing value: $\left(\frac{3}{2}\right)^2=\frac{9}{4}$.
Step 3: Add $\frac{9}{4}$ inside. Compensation: $4\times\frac{9}{4}=9$. Subtract 9 outside.

$$4\left(x^2+3x+\frac{9}{4}\right)-1-9$$

Step 4: Rewrite. $x^2+3x+\frac{9}{4}=\left(x+\frac{3}{2}\right)^2$.

$$4\left(x+\frac{3}{2}\right)^2-1-9$$

Step 5:

$$4\left(x+\frac{3}{2}\right)^2-10$$

$$\boxed{4\left(x+\frac{3}{2}\right)^2-10}$$

11. (Challenge) $\frac{1}{3}x^2-2x+5$

Step 1: $a=\frac{1}{3}$. Factor $\frac{1}{3}$ out of the first two terms.

$$\frac{1}{3}(x^2-6x)+5$$

Step 2: Coefficient of x inside is -6. Completing value: $\left(\frac{-6}{2}\right)^2=9$.

Step 3: Add 9 inside. Compensation: $\frac{1}{3}\times 9=3$. Subtract 3 outside.

$$\frac{1}{3}(x^2-6x+9)+5-3$$

Step 4:

$$\frac{1}{3}(x-3)^2+5-3$$

Step 5:

$$\frac{1}{3}(x-3)^2+2$$

$$\boxed{\frac{1}{3}(x-3)^2+2}$$

12. (Challenge) $-\frac{3}{2}x^2+6x-1$

Step 1: $a=-\frac{3}{2}$. Factor $-\frac{3}{2}$ out of the first two terms.

$$-\frac{3}{2}(x^2-4x)-1$$

Step 2: Coefficient of x inside is -4. Completing value: $\left(\frac{-4}{2}\right)^2=4$.

Step 3: Add 4 inside. Compensation: $\left(-\frac{3}{2}\right)\times 4=-6$. Subtract -6 (add 6) outside.

$$-\frac{3}{2}(x^2-4x+4)-1+6$$

Step 4:

$$-\frac{3}{2}(x-2)^2-1+6$$

Step 5:

$$-\frac{3}{2}(x-2)^2+5$$

$$\boxed{-\frac{3}{2}(x-2)^2+5}$$

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