The Quadratic Formula
ApexMath

Notes

What Is the Quadratic Formula?

The quadratic formula solves *any* quadratic equation of the form $ax^2 + bx + c = 0$, regardless of whether it factors nicely. The formula is:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The $\pm$ symbol means you get two answers: one using $+$ and one using $-$.

Where Does the Formula Come From?

The quadratic formula is derived by completing the square on the general equation $ax^2 + bx + c = 0$. Every step is shown below so you can see exactly why the formula is true.

Start with the general equation:

$$ax^2 + bx + c = 0$$

Subtract c from both sides:

$$ax^2 + bx = -c$$

Divide every term by a (since $a \neq 0$):

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square on the left side. The completing value is $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$. Add it to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Rewrite the left side as a squared binomial:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Rewrite the right side with a common denominator of $4a^2$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides. Remember to include $\pm$:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Combine over the common denominator $2a$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant

The expression under the square root, $b^2 - 4ac$, is called the **discriminant**. It tells you how many real solutions the equation has *before* you solve it.

Discriminant	Number of Solutions	What It Means
$b^2 - 4ac > 0$	Two distinct real solutions	Parabola crosses x-axis twice
$b^2 - 4ac = 0$	One repeated real solution	Parabola touches x-axis once
$b^2 - 4ac < 0$	No real solutions	Parabola does not cross x-axis

How to Apply the Formula

1. Write the equation in standard form $ax^2 + bx + c = 0$.
2. Identify a, b, and c.
3. Compute the discriminant $b^2 - 4ac$.
4. Substitute into the formula and simplify.
5. Simplify the square root if possible.

Pro Tip

- Always move all terms to one side and write the equation equal to zero before identifying a, b, and c. Reading a, b, c from an equation that is not set to zero is the most common way to get the wrong answer entirely.
- Compute the discriminant first as its own step. Knowing whether it is positive, zero, or negative tells you exactly how many answers to expect before you simplify the full formula.
- Keep the $\pm$ in your work until the very last step. It is a common mistake to drop one of the two solutions by forgetting to evaluate both $+$ and $-$.
- The denominator is $2a$, not 2 and not a. Every part of the numerator — including $-b$ — is divided by $2a$.
- Simplify the square root before splitting into two answers. For example, $\sqrt{12} = 2\sqrt{3}$ first, then separate into $\frac{-b + 2\sqrt{3}}{2a}$ and $\frac{-b - 2\sqrt{3}}{2a}$.

Examples

Example 1 (Two rational solutions): Solve $x^2 - 5x + 6 = 0$.

Step 1: The equation is already in standard form.
Step 2: Identify $a = 1$, $b = -5$, $c = 6$.
Step 3: Compute the discriminant.

$$b^2 - 4ac = (-5)^2 - 4(1)(6) = 25 - 24 = 1$$

The discriminant is positive, so there are two distinct real solutions.
Step 4: Substitute into the quadratic formula.

$$x = \frac{-(-5) \pm \sqrt{1}}{2(1)} = \frac{5 \pm 1}{2}$$

Step 5: Split into two solutions.

$$x = \frac{5+1}{2} = \frac{6}{2} = 3 \qquad \text{or} \qquad x = \frac{5-1}{2} = \frac{4}{2} = 2$$

$$\boxed{x = 3 \quad \text{or} \quad x = 2}$$

Example 2 (Two irrational solutions): Solve $x^2 + 4x + 1 = 0$.

Step 1: Standard form confirmed.
Step 2: $a = 1$, $b = 4$, $c = 1$.
Step 3: Discriminant.

$$b^2 - 4ac = 4^2 - 4(1)(1) = 16 - 4 = 12$$

Positive, so two distinct real solutions.
Step 4: Substitute.

$$x = \frac{-4 \pm \sqrt{12}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2}$$

Step 5: Simplify $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$.

$$x = \frac{-4 \pm 2\sqrt{3}}{2}$$

Divide every term in the numerator by 2:

$$x = -2 \pm \sqrt{3}$$

$$\boxed{x = -2 + \sqrt{3} \quad \text{or} \quad x = -2 - \sqrt{3}}$$

Example 3 (One repeated solution): Solve $x^2 - 6x + 9 = 0$.

Step 1: Standard form confirmed.
Step 2: $a = 1$, $b = -6$, $c = 9$.
Step 3: Discriminant.

$$b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$

The discriminant is zero, so there is exactly one repeated solution.
Step 4: Substitute.

$$x = \frac{-(-6) \pm \sqrt{0}}{2(1)} = \frac{6 \pm 0}{2} = \frac{6}{2} = 3$$

$$\boxed{x = 3 \text{ (repeated root)}}$$

Example 4 (No real solutions): Solve $x^2 + 2x + 5 = 0$.

Step 1: Standard form confirmed.
Step 2: $a = 1$, $b = 2$, $c = 5$.
Step 3: Discriminant.

$$b^2 - 4ac = 2^2 - 4(1)(5) = 4 - 20 = -16$$

The discriminant is negative, so there are **no real solutions**.
The parabola $y = x^2 + 2x + 5$ does not cross the x-axis.

$$\boxed{\text{No real solutions}}$$

Example 5 ($a \neq 1$, equation needs rearranging): Solve $2x^2 + 3x - 2 = 0$.

Step 1: Standard form confirmed.
Step 2: $a = 2$, $b = 3$, $c = -2$.
Step 3: Discriminant.

$$b^2 - 4ac = 3^2 - 4(2)(-2) = 9 + 16 = 25$$

Positive, so two distinct real solutions.
Step 4: Substitute.

$$x = \frac{-3 \pm \sqrt{25}}{2(2)} = \frac{-3 \pm 5}{4}$$

Step 5: Split into two solutions.

$$x = \frac{-3 + 5}{4} = \frac{2}{4} = \frac{1}{2} \qquad \text{or} \qquad x = \frac{-3 - 5}{4} = \frac{-8}{4} = -2$$

$$\boxed{x = \frac{1}{2} \quad \text{or} \quad x = -2}$$

Example 6 (Rearrange first): Solve $3x^2 = 7x - 2$.

Step 1: Move all terms to one side. Subtract $7x$ and add 2 to both sides.

$$3x^2 - 7x + 2 = 0$$

Step 2: $a = 3$, $b = -7$, $c = 2$.
Step 3: Discriminant.

$$b^2 - 4ac = (-7)^2 - 4(3)(2) = 49 - 24 = 25$$

Positive, so two distinct real solutions.
Step 4: Substitute.

$$x = \frac{-(-7) \pm \sqrt{25}}{2(3)} = \frac{7 \pm 5}{6}$$

Step 5: Split.

$$x = \frac{7+5}{6} = \frac{12}{6} = 2 \qquad \text{or} \qquad x = \frac{7-5}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\boxed{x = 2 \quad \text{or} \quad x = \frac{1}{3}}$$

Common Mistakes

- **Not writing the equation equal to zero first:** You must have $ax^2 + bx + c = 0$ before reading off a, b, and c. If you read them from an equation like $3x^2 + x = 5$, you will use the wrong value of c.

- **Dropping the negative on b:** The formula begins with $-b$. If $b = -5$, then $-b = -(-5) = 5$. Forgetting to flip the sign of b is an extremely common arithmetic error.

- **Only dividing part of the numerator by $2a$:** The entire numerator, $-b \pm \sqrt{b^2 - 4ac}$, is divided by $2a$. A common error is to write $\frac{-b}{2a} \pm \sqrt{b^2 - 4ac}$ instead of putting the whole expression over $2a$.

- **Computing $4ac$ incorrectly when c is negative:** If $c = -3$ and $a = 2$, then $4ac = 4(2)(-3) = -24$, and $b^2 - 4ac = b^2 - (-24) = b^2 + 24$. Watch the sign carefully when c is negative — subtracting a negative adds.

- **Forgetting to simplify the square root:** Always check whether $\sqrt{b^2 - 4ac}$ simplifies. For example, $\sqrt{12}$ should be written as $2\sqrt{3}$ before giving the final answer.

Worksheet

For each problem, identify a, b, c, compute the discriminant, and solve using the quadratic formula. Simplify all answers fully.

1. $x^2 + 7x + 10 = 0$

2. $x^2 - 3x - 4 = 0$

3. $x^2 + 6x + 9 = 0$

4. $x^2 - 4x + 6 = 0$

5. $2x^2 - 7x + 3 = 0$

6. $3x^2 + x - 2 = 0$

7. $x^2 + 2x - 4 = 0$

8. $4x^2 - 4x + 1 = 0$

9. $2x^2 + 5x = 3$

10. (Challenge) $6x^2 - x - 2 = 0$

11. (Challenge) $x^2 - 2x - 7 = 0$

12. (Challenge) Without solving, determine how many real solutions $5x^2 - 3x + 1 = 0$ has. Justify your answer using the discriminant.

Answer Key

1. $x^2 + 7x + 10 = 0$
   Step 1: $a = 1$, $b = 7$, $c = 10$.
   Step 2: Discriminant.

$$b^2 - 4ac = 7^2 - 4(1)(10) = 49 - 40 = 9$$

Two real solutions.
   Step 3: Apply the formula.

$$x = \frac{-7 \pm \sqrt{9}}{2(1)} = \frac{-7 \pm 3}{2}$$

   Step 4: Split.

$$x = \frac{-7+3}{2} = \frac{-4}{2} = -2 \qquad \text{or} \qquad x = \frac{-7-3}{2} = \frac{-10}{2} = -5$$

$$\boxed{x = -2 \quad \text{or} \quad x = -5}$$

2. $x^2 - 3x - 4 = 0$
Step 1: $a = 1$, $b = -3$, $c = -4$.
Step 2: Discriminant.

$$b^2 - 4ac = (-3)^2 - 4(1)(-4) = 9 + 16 = 25$$

Two real solutions.

Step 3:

$$x = \frac{-(-3) \pm \sqrt{25}}{2(1)} = \frac{3 \pm 5}{2}$$

Step 4:

$$x = \frac{3+5}{2} = \frac{8}{2} = 4 \qquad \text{or} \qquad x = \frac{3-5}{2} = \frac{-2}{2} = -1$$

$$\boxed{x = 4 \quad \text{or} \quad x = -1}$$

3. $x^2 + 6x + 9 = 0$

Step 1: $a = 1$, $b = 6$, $c = 9$.

Step 2: Discriminant.

$$b^2 - 4ac = 6^2 - 4(1)(9) = 36 - 36 = 0$$

One repeated solution.

Step 3:

$$x = \frac{-6 \pm \sqrt{0}}{2(1)} = \frac{-6}{2} = -3$$

$$\boxed{x = -3 \text{ (repeated root)}}$$

4. $x^2 - 4x + 6 = 0$

Step 1: $a = 1$, $b = -4$, $c = 6$.

Step 2: Discriminant.

$$b^2 - 4ac = (-4)^2 - 4(1)(6) = 16 - 24 = -8$$

Discriminant is negative. No real solutions.

$$\boxed{\text{No real solutions}}$$

5. $2x^2 - 7x + 3 = 0$

Step 1: $a = 2$, $b = -7$, $c = 3$.

Step 2: Discriminant.

$$b^2 - 4ac = (-7)^2 - 4(2)(3) = 49 - 24 = 25$$

Two real solutions.

Step 3:

$$x = \frac{-(-7) \pm \sqrt{25}}{2(2)} = \frac{7 \pm 5}{4}$$

Step 4:

$$x = \frac{7+5}{4} = \frac{12}{4} = 3 \qquad \text{or} \qquad x = \frac{7-5}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{x = 3 \quad \text{or} \quad x = \frac{1}{2}}$$

6. $3x^2 + x - 2 = 0$
Step 1: $a = 3$, $b = 1$, $c = -2$.
Step 2: Discriminant.

$$b^2 - 4ac = 1^2 - 4(3)(-2) = 1 + 24 = 25$$

Two real solutions.
Step 3:

$$x = \frac{-1 \pm \sqrt{25}}{2(3)} = \frac{-1 \pm 5}{6}$$

Step 4:

$$x = \frac{-1+5}{6} = \frac{4}{6} = \frac{2}{3} \qquad \text{or} \qquad x = \frac{-1-5}{6} = \frac{-6}{6} = -1$$

$$\boxed{x = \frac{2}{3} \quad \text{or} \quad x = -1}$$

7. $x^2 + 2x - 4 = 0$
Step 1: $a = 1$, $b = 2$, $c = -4$.
Step 2: Discriminant.

$$b^2 - 4ac = 2^2 - 4(1)(-4) = 4 + 16 = 20$$

Two real solutions.
Step 3:

$$x = \frac{-2 \pm \sqrt{20}}{2(1)} = \frac{-2 \pm \sqrt{20}}{2}$$

Step 4: Simplify $\sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$.

$$x = \frac{-2 \pm 2\sqrt{5}}{2}$$

Divide every term in the numerator by 2:

$$x = -1 \pm \sqrt{5}$$

$$\boxed{x = -1 + \sqrt{5} \quad \text{or} \quad x = -1 - \sqrt{5}}$$

8. $4x^2 - 4x + 1 = 0$
Step 1: $a = 4$, $b = -4$, $c = 1$.

Step 2: Discriminant.

$$b^2 - 4ac = (-4)^2 - 4(4)(1) = 16 - 16 = 0$$

One repeated solution.
Step 3:

$$x = \frac{-(-4) \pm \sqrt{0}}{2(4)} = \frac{4}{8} = \frac{1}{2}$$

$$\boxed{x = \frac{1}{2} \text{ (repeated root)}}$$

9. $2x^2 + 5x = 3$
Step 1: Move all terms to one side. Subtract 3 from both sides.

$$2x^2 + 5x - 3 = 0$$

Step 2: $a = 2$, $b = 5$, $c = -3$.
Step 3: Discriminant.

$$b^2 - 4ac = 5^2 - 4(2)(-3) = 25 + 24 = 49$$

Two real solutions.
Step 4:

$$x = \frac{-5 \pm \sqrt{49}}{2(2)} = \frac{-5 \pm 7}{4}$$

Step 5:

$$x = \frac{-5+7}{4} = \frac{2}{4} = \frac{1}{2} \qquad \text{or} \qquad x = \frac{-5-7}{4} = \frac{-12}{4} = -3$$

$$\boxed{x = \frac{1}{2} \quad \text{or} \quad x = -3}$$

10. (Challenge) $6x^2 - x - 2 = 0$
Step 1: $a = 6$, $b = -1$, $c = -2$.
Step 2: Discriminant.

$$b^2 - 4ac = (-1)^2 - 4(6)(-2) = 1 + 48 = 49$$

Two real solutions.
Step 3:

$$x = \frac{-(-1) \pm \sqrt{49}}{2(6)} = \frac{1 \pm 7}{12}$$

Step 4:

$$x = \frac{1+7}{12} = \frac{8}{12} = \frac{2}{3} \qquad \text{or} \qquad x = \frac{1-7}{12} = \frac{-6}{12} = -\frac{1}{2}$$

$$\boxed{x = \frac{2}{3} \quad \text{or} \quad x = -\frac{1}{2}}$$

11. (Challenge) $x^2 - 2x - 7 = 0$
Step 1: $a = 1$, $b = -2$, $c = -7$.
Step 2: Discriminant.

$$b^2 - 4ac = (-2)^2 - 4(1)(-7) = 4 + 28 = 32$$

Two real solutions.
Step 3:

$$x = \frac{-(-2) \pm \sqrt{32}}{2(1)} = \frac{2 \pm \sqrt{32}}{2}$$

Step 4: Simplify $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$.

$$x = \frac{2 \pm 4\sqrt{2}}{2}$$

Divide every term in the numerator by 2:

$$x = 1 \pm 2\sqrt{2}$$

$$\boxed{x = 1 + 2\sqrt{2} \quad \text{or} \quad x = 1 - 2\sqrt{2}}$$

12. (Challenge) Without solving, determine how many real solutions $5x^2 - 3x + 1 = 0$ has.
Step 1: $a = 5$, $b = -3$, $c = 1$.
Step 2: Compute the discriminant only.

$$b^2 - 4ac = (-3)^2 - 4(5)(1) = 9 - 20 = -11$$

Step 3: Since the discriminant is $-11 < 0$, there are **no real solutions**. A negative number has no real square root, so the quadratic formula produces no real values of x.

Geometrically, the parabola $y = 5x^2 - 3x + 1$ opens upward (since $a = 5 > 0$) and its vertex lies above the x-axis, so it never crosses it.

$$\boxed{\text{No real solutions} \quad (b^2 - 4ac = -11 < 0)}$$

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End of Lesson • ApexMath
