Dividing Radicals and Rationalizing the Denominator
ApexMath

Notes

The Big Idea

Just like multiplication, division of radicals follows a clean rule:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

You can divide the numbers inside the radicals, or you can simplify the fraction inside the radical. Either approach leads to the same answer.

However, there is one important rule in mathematics:

A radical in the denominator of a fraction is not considered simplified.

The process of removing a radical from the denominator is called **rationalizing the denominator**. This is the main skill in this lesson.

Case 1 — Simple Division

When the radicands divide evenly, simplify the fraction inside the radical first.

$$\frac{\sqrt{50}}{\sqrt{2}} = \sqrt{\frac{50}{2}} = \sqrt{25} = 5$$

Always check this first. If the radicands divide evenly, this is the fastest approach.

Case 2 — Rationalizing with a Simple Radical Denominator

When the denominator is a single radical and you cannot divide evenly, multiply both the numerator and denominator by that same radical. Since you are multiplying by $\frac{\sqrt{b}}{\sqrt{b}} = 1$, the value of the fraction does not change.

$$\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{5}}{5}$$

The denominator is now a whole number. The fraction is rationalized.

Case 3 — Rationalizing with a Binomial Denominator (Conjugates)

When the denominator contains two terms — one of which involves a radical, such as $(3 + \sqrt{2})$ — you cannot simply multiply by the radical alone. Instead, you multiply by the **conjugate**.

The conjugate of $(a + \sqrt{b})$ is $(a - \sqrt{b})$, and vice versa. You simply flip the sign between the two terms.

- Conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$
- Conjugate of $(1 - \sqrt{5})$ is $(1 + \sqrt{5})$

When you multiply a binomial by its conjugate, the radical always disappears from the denominator:

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

This is the difference of squares pattern you saw in the previous lesson.

The Full Method for Rationalizing a Binomial Denominator

- Step 1: Identify the conjugate of the denominator.
- Step 2: Multiply both the numerator and denominator by that conjugate.
- Step 3: Expand the denominator using the difference of squares pattern.
- Step 4: Expand the numerator by distributing.
- Step 5: Simplify the result completely. Reduce any common factors if possible.

Pro tip

- Always check Case 1 first. If the radicands divide evenly, you can avoid rationalizing altogether.
- When rationalizing, you are multiplying by a form of 1, so the value never changes — only the appearance of the fraction changes.
- The conjugate only flips the sign between the two terms. Do not change the numbers or the radical itself.
- After rationalizing, always simplify the fraction fully. Check whether the numerator and denominator share a common factor that can be divided out.
- The denominator after using conjugates will always be a whole number. If you still have a radical in your denominator, something went wrong.

Examples

Example A

Simplify $\dfrac{\sqrt{72}}{\sqrt{8}}$.

Step 1: Check whether the radicands divide evenly.

$$\frac{72}{8} = 9$$

Step 2: Simplify inside the radical.

$$\frac{\sqrt{72}}{\sqrt{8}} = \sqrt{\frac{72}{8}} = \sqrt{9} = 3$$

Example B

Rationalize the denominator of $\frac{5}{\sqrt{3}}$.

Step 1: The denominator is $\sqrt{3}$, a single radical. Multiply the numerator and denominator by $\sqrt{3}$.

$$\frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

Step 2: Multiply out.

$$\frac{5\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{5\sqrt{3}}{3}$$

Step 3: Check whether the fraction simplifies. 5 and 3 share no common factors.

$$\frac{5\sqrt{3}}{3}$$

Example C

Rationalize the denominator of $\frac{6}{\sqrt{12}}$.

Step 1: Simplify $\sqrt{12}$ first.

$$\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$$

Step 2: Rewrite the fraction.

$$\frac{6}{2\sqrt{3}}$$

Step 3: Simplify the coefficient part.

$$\frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Step 4: Multiply numerator and denominator by $\sqrt{3}$.

$$\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Example D

Rationalize the denominator of $\frac{4}{3+\sqrt{2}}$.

Step 1: Identify the conjugate of the denominator.

$$\text{Conjugate of } (3+\sqrt{2}) \text{ is } (3-\sqrt{2})$$

Step 2: Multiply both numerator and denominator by the conjugate.

$$\frac{4}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}}$$

Step 3: Expand the denominator using the difference of squares pattern.

$$(3+\sqrt{2})(3-\sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7$$

Step 4: Expand the numerator.

$$4 \times (3-\sqrt{2}) = 12 - 4\sqrt{2}$$

Step 5: Write the rationalized fraction.

$$\frac{12-4\sqrt{2}}{7}$$

Step 6: Check whether anything simplifies. 12, 4, and 7 share no common factors, so this is the final answer.

$$\frac{12-4\sqrt{2}}{7}$$

Example E

Rationalize the denominator of $\dfrac{3+\sqrt{5}}{2-\sqrt{5}}$.

Step 1: The conjugate of $(2-\sqrt{5})$ is $(2+\sqrt{5})$.
Step 2: Multiply both numerator and denominator by the conjugate.

$$\frac{3+\sqrt{5}}{2-\sqrt{5}} \times \frac{2+\sqrt{5}}{2+\sqrt{5}}$$

Step 3: Expand the denominator.

$$(2-\sqrt{5})(2+\sqrt{5}) = 4 - 5 = -1$$

Step 4: Expand the numerator.

$$(3+\sqrt{5})(2+\sqrt{5}) = 3 \times 2 + 3 \times \sqrt{5} + \sqrt{5} \times 2 + \sqrt{5} \times \sqrt{5} = 6 + 3\sqrt{5} + 2\sqrt{5} + 5 = 11 + 5\sqrt{5}$$

Step 5: Write the fraction.

$$\frac{11+5\sqrt{5}}{-1}$$

Step 6: Divide each term in the numerator by -1.

$$-11 - 5\sqrt{5}$$

Common mistakes

- **Only multiplying the denominator by the radical.** You must multiply **both** the numerator and the denominator. You are multiplying by a form of 1 — if you only change the denominator, you change the value of the fraction.

- **Using the wrong conjugate.** The conjugate only changes the sign between the two terms. The conjugate of $(5-\sqrt{3})$ is $(5+\sqrt{3})$, not $(-5+\sqrt{3})$.

- **Expanding the denominator incorrectly.** $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$, not $a^2 - \sqrt{b^2}$. The square root disappears entirely.

- **Forgetting to simplify after rationalizing.** Always check whether the resulting fraction reduces. If the numerator and denominator share a common factor, divide it out.

Worksheet

1. Simplify each expression by dividing inside the radical.

   (a) $\dfrac{\sqrt{100}}{\sqrt{4}}$

   (b) $\dfrac{\sqrt{75}}{\sqrt{3}}$

   (c) $\dfrac{\sqrt{98}}{\sqrt{2}}$

   (d) $\dfrac{\sqrt{200}}{\sqrt{8}}$

2. Rationalize the denominator of each expression.

(a) $\dfrac{1}{\sqrt{2}}$

(b) $\dfrac{4}{\sqrt{7}}$

(c) $\dfrac{10}{\sqrt{5}}$

(d) $\dfrac{3}{\sqrt{6}}$

3. Simplify first, then rationalize the denominator.

(a) $\dfrac{8}{\sqrt{16}}$

(b) $\dfrac{6}{\sqrt{18}}$

(c) $\dfrac{10}{\sqrt{50}}$

(d) $\dfrac{12}{\sqrt{48}}$

4. Write the conjugate of each expression.

(a) $2+\sqrt{3}$

(b) $5-\sqrt{7}$

(c) $\sqrt{5}+4$

(d) $\sqrt{6}-\sqrt{2}$

5. Rationalize the denominator of each expression.

(a) $\dfrac{3}{1+\sqrt{2}}$

(b) $\dfrac{5}{3-\sqrt{2}}$

(c) $\dfrac{2}{4+\sqrt{3}}$

6. Rationalize the denominator of $\dfrac{1+\sqrt{3}}{2-\sqrt{3}}$.

7. Rationalize the denominator of $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$.

8. A student rationalized $\dfrac{6}{2+\sqrt{2}}$ and got $\dfrac{6(2-\sqrt{2})}{2}$. Identify and correct the mistake.

9. **Challenge:** Rationalize the denominator of $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$.

10. **Challenge:** Rationalize the denominator of $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

Answer Key

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1.

(a) $\dfrac{\sqrt{100}}{\sqrt{4}}$

Step 1: Divide the radicands.

$$\frac{100}{4} = 25$$

Step 2: Evaluate.

$$\sqrt{25} = 5$$

$$\boxed{5}$$

(b) $\dfrac{\sqrt{75}}{\sqrt{3}}$

Step 1: Divide the radicands.

$$\frac{75}{3} = 25$$

Step 2: Evaluate.

$$\sqrt{25} = 5$$

$$\boxed{5}$$

(c) $\dfrac{\sqrt{98}}{\sqrt{2}}$

Step 1: Divide the radicands.

$$\frac{98}{2} = 49$$

Step 2: Evaluate.

$$\sqrt{49} = 7$$

$$\boxed{7}$$

(d) $\dfrac{\sqrt{200}}{\sqrt{8}}$

Step 1: Divide the radicands.

$$\frac{200}{8} = 25$$

Step 2: Evaluate.

$$\sqrt{25} = 5$$

$$\boxed{5}$$

2.

(a) $\dfrac{1}{\sqrt{2}}$

Step 1: Multiply numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\boxed{\frac{\sqrt{2}}{2}}$$

(b) $\dfrac{4}{\sqrt{7}}$

Step 1: Multiply numerator and denominator by $\sqrt{7}$.

$$\frac{4}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$$

$$\boxed{\frac{4\sqrt{7}}{7}}$$

(c) $\dfrac{10}{\sqrt{5}}$

Step 1: Multiply numerator and denominator by $\sqrt{5}$.

$$\frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

$$\boxed{2\sqrt{5}}$$

(d) $\dfrac{3}{\sqrt{6}}$

Step 1: Multiply numerator and denominator by $\sqrt{6}$.

$$\frac{3}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$$

$$\boxed{\frac{\sqrt{6}}{2}}$$

3.

(a) $\dfrac{8}{\sqrt{16}}$

Step 1: Evaluate $\sqrt{16} = 4$.

$$\frac{8}{4} = 2$$

$$\boxed{2}$$

(b) $\dfrac{6}{\sqrt{18}}$

Step 1: Simplify $\sqrt{18}$.

$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$

Step 2: Rewrite the fraction.

$$\frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}}$$

Step 3: Multiply numerator and denominator by $\sqrt{2}$.

$$\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\boxed{\sqrt{2}}$$

(c) $\dfrac{10}{\sqrt{50}}$

Step 1: Simplify $\sqrt{50}$.

$$\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$$

Step 2: Rewrite the fraction.

$$\frac{10}{5\sqrt{2}} = \frac{2}{\sqrt{2}}$$

Step 3: Multiply numerator and denominator by $\sqrt{2}$.

$$\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\boxed{\sqrt{2}}$$

(d) $\dfrac{12}{\sqrt{48}}$

Step 1: Simplify $\sqrt{48}$.

$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$

Step 2: Rewrite the fraction.

$$\frac{12}{4\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Step 3: Multiply numerator and denominator by $\sqrt{3}$.

$$\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$\boxed{\sqrt{3}}$$

4.

(a) Conjugate of $2 + \sqrt{3}$
Step 1: Flip the sign between the two terms.

$$\boxed{2 - \sqrt{3}}$$

(b) Conjugate of $5 - \sqrt{7}$
Step 1: Flip the sign between the two terms.

$$\boxed{5 + \sqrt{7}}$$

(c) Conjugate of $\sqrt{5} + 4$
Step 1: Flip the sign between the two terms.

$$\boxed{\sqrt{5} - 4}$$

(d) Conjugate of $\sqrt{6} - \sqrt{2}$
Step 1: Flip the sign between the two terms.

$$\boxed{\sqrt{6} + \sqrt{2}}$$

5.

(a) $\dfrac{3}{1 + \sqrt{2}}$
Step 1: The conjugate of $(1 + \sqrt{2})$ is $(1 - \sqrt{2})$.
Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{3}{1 + \sqrt{2}} \times \frac{1 - \sqrt{2}}{1 - \sqrt{2}}$$

Step 3: Expand the denominator.

$$(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$$

Step 4: Expand the numerator.

$$3(1 - \sqrt{2}) = 3 - 3\sqrt{2}$$

Step 5: Write the fraction and divide by -1.

$$\frac{3 - 3\sqrt{2}}{-1} = -3 + 3\sqrt{2}$$

$$\boxed{3\sqrt{2} - 3}$$

(b) $\dfrac{5}{3-\sqrt{2}}$

Step 1: The conjugate of $(3-\sqrt{2})$ is $(3+\sqrt{2})$.

Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{5}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

Step 3: Expand the denominator.

$$(3-\sqrt{2})(3+\sqrt{2}) = 9-2 = 7$$

Step 4: Expand the numerator.

$$5(3+\sqrt{2}) = 15+5\sqrt{2}$$

Step 5: Write the final fraction.

$$\frac{15+5\sqrt{2}}{7}$$

$$\boxed{\frac{15+5\sqrt{2}}{7}}$$

(c) $\dfrac{2}{4+\sqrt{3}}$

Step 1: The conjugate of $(4+\sqrt{3})$ is $(4-\sqrt{3})$.

Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{2}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}}$$

Step 3: Expand the denominator.

$$(4+\sqrt{3})(4-\sqrt{3}) = 16-3 = 13$$

Step 4: Expand the numerator.

$$2(4-\sqrt{3}) = 8-2\sqrt{3}$$

Step 5: Write the final fraction.

$$\frac{8-2\sqrt{3}}{13}$$

$$\boxed{\frac{8-2\sqrt{3}}{13}}$$

6. Rationalize $\dfrac{1+\sqrt{3}}{2-\sqrt{3}}$.

Step 1: The conjugate of $(2-\sqrt{3})$ is $(2+\sqrt{3})$.
Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{1+\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

Step 3: Expand the denominator.

$$(2-\sqrt{3})(2+\sqrt{3}) = 4-3 = 1$$

Step 4: Expand the numerator.

$$(1+\sqrt{3})(2+\sqrt{3}) = 1\times 2 + 1\times\sqrt{3} + \sqrt{3}\times 2 + \sqrt{3}\times\sqrt{3} = 2+\sqrt{3}+2\sqrt{3}+3 = 5+3\sqrt{3}$$

Step 5: Write the fraction. Since the denominator is 1, write the numerator directly.

$$\boxed{5+3\sqrt{3}}$$

7. Rationalize $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$.
Step 1: The conjugate of $(\sqrt{5}-1)$ is $(\sqrt{5}+1)$.
Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{\sqrt{5}+1}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}$$

Step 3: Expand the denominator.

$$(\sqrt{5}-1)(\sqrt{5}+1) = 5-1 = 4$$

Step 4: Expand the numerator.

$$(\sqrt{5}+1)(\sqrt{5}+1) = \sqrt{5}\times\sqrt{5} + \sqrt{5}\times 1 + 1\times\sqrt{5} + 1\times 1 = 5+\sqrt{5}+\sqrt{5}+1 = 6+2\sqrt{5}$$

Step 5: Write the fraction.

$$\frac{6+2\sqrt{5}}{4}$$

Step 6: Divide every term in the numerator by 2 to reduce.

$$\frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$\boxed{\frac{3+\sqrt{5}}{2}}$$

8. A student rationalized $\dfrac{6}{2+\sqrt{2}}$ and got $\dfrac{6(2-\sqrt{2})}{2}$. Find and correct the mistake.

Step 1: Identify the error. The student expanded the denominator incorrectly. They wrote the denominator as 2 instead of computing $(2+\sqrt{2})(2-\sqrt{2})$ properly.

Step 2: Compute the denominator correctly.

$$(2+\sqrt{2})(2-\sqrt{2}) = 4-2 = 2$$

Step 3: Actually, the denominator is 2, so the student's denominator was coincidentally correct in value. However, the student likely skipped showing the expansion. Let us verify the full solution.

Step 4: Expand the numerator.

$$6(2-\sqrt{2}) = 12-6\sqrt{2}$$

Step 5: Write the fraction.

$$\frac{12-6\sqrt{2}}{2}$$

Step 6: Divide every term by 2 to reduce fully.

$$\frac{12-6\sqrt{2}}{2} = 6-3\sqrt{2}$$

Step 7: State the mistake. The student stopped too early and did not reduce the fraction. The fully simplified answer is:

$$\boxed{6-3\sqrt{2}}$$

9. Rationalize $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$.

Step 1: The conjugate of $(3-\sqrt{2})$ is $(3+\sqrt{2})$.

Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{3+\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

Step 3: Expand the denominator.

$$(3-\sqrt{2})(3+\sqrt{2}) = 9-2 = 7$$

Step 4: Expand the numerator.

$$(3+\sqrt{2})(3+\sqrt{2}) = 3\times 3+3\times\sqrt{2}+\sqrt{2}\times 3+\sqrt{2}\times\sqrt{2} = 9+3\sqrt{2}+3\sqrt{2}+2 = 11+6\sqrt{2}$$

Step 5: Write the final fraction.

$$\boxed{\frac{11+6\sqrt{2}}{7}}$$

10. Rationalize $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

Step 1: The conjugate of $(\sqrt{6}-\sqrt{2})$ is $(\sqrt{6}+\sqrt{2})$.
Step 2: Multiply numerator and denominator by the conjugate.

$$\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

Step 3: Expand the denominator.

$$(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2}) = 6-2 = 4$$

Step 4: Expand the numerator.

$$(\sqrt{6}+\sqrt{2})(\sqrt{6}+\sqrt{2}) = \sqrt{6}\times\sqrt{6}+\sqrt{6}\times\sqrt{2}+\sqrt{2}\times\sqrt{6}+\sqrt{2}\times\sqrt{2}$$

$$= 6+\sqrt{12}+\sqrt{12}+2$$

Step 5: Simplify $\sqrt{12}$.

$$\sqrt{12} = \sqrt{4\times 3} = 2\sqrt{3}$$

Step 6: Substitute back.

$$6+2\sqrt{3}+2\sqrt{3}+2 = 8+4\sqrt{3}$$

Step 7: Write the fraction.

$$\frac{8+4\sqrt{3}}{4}$$

Step 8: Divide every term by 4 to reduce.

$$\frac{8+4\sqrt{3}}{4} = 2+\sqrt{3}$$

$$\boxed{2+\sqrt{3}}$$

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