Extraneous Roots and Domain Restrictions
ApexMath

Notes

The Big Idea

In the previous lesson, you learned that checking your answer is mandatory when solving radical equations. The reason is that squaring both sides of an equation can produce a solution that fits the algebra but not the original equation. These false solutions are called **extraneous roots**.

This lesson goes deeper. You will learn *why* extraneous roots appear, how to predict when they might occur, and how the concept of **domain** connects to all of this.

What is a Domain?

The **domain** of an expression is the set of all values of x that are allowed — that is, values for which the expression is defined and produces a real number.

For radical expressions involving square roots, there is one key restriction:

The expression inside a square root cannot be negative.

This is because there is no real number whose square is negative. For example, $\sqrt{-9}$ is not a real number.

So for an expression like $\sqrt{x+3}$, we need:

$$x + 3 \geq 0 \qquad \Rightarrow \qquad x \geq -3$$

This means the domain of $\sqrt{x+3}$ is all values $x \geq -3$.

How to Find the Domain of a Radical Expression

- Step 1: Set the radicand greater than or equal to zero.
- Step 2: Solve the inequality.
- Step 3: State the domain using inequality notation or interval notation.

For example, find the domain of $\sqrt{2x-10}$:

$$2x - 10 \geq 0$$

$$2x \geq 10$$

$$x \geq 5$$

The domain is all real numbers $x \geq 5$.

Why Extraneous Roots Appear

When you solve a radical equation by squaring both sides, you are solving a broader equation than the original. Squaring removes the sign restriction on the radical. This means values that were outside the domain can slip in as apparent solutions.

Consider: $\sqrt{x} = -3$

The domain of $\sqrt{x}$ requires $x \geq 0$. Also, $\sqrt{x}$ is always non-negative, so it can never equal -3. But if you square both sides without thinking:

$$x = 9$$

Substituting back: $\sqrt{9} = 3 \neq -3$. So $x = 9$ is extraneous. The domain and the sign of the radical both told us this would happen before we even started.

Checking Domain Restrictions Before Solving

A smart habit is to check the domain restriction *before* solving. This lets you immediately identify whether certain solutions are impossible.

For the equation $\sqrt{x+5} = x - 1$:

- The left side requires $x + 5 \geq 0$, so $x \geq -5$.
- The left side $\sqrt{x+5}$ is always ≥ 0, so the right side $x - 1$ must also be ≥ 0, meaning $x \geq 1$.

Any solution with $x < 1$ will be extraneous before we even check it.

Cube Roots Have No Domain Restriction

Unlike square roots, cube roots are defined for all real numbers — positive, negative, and zero. This is because a negative number cubed is negative:

$$(-2)^3 = -8 \qquad \text{so} \qquad \sqrt[3]{-8} = -2$$

Therefore, equations involving cube roots do not produce extraneous roots from squaring, and there are no domain restrictions on the radicand.

Summarizing When to Expect Extraneous Roots

Extraneous roots are most likely when:

- The right-hand side of the equation is negative (a square root can never equal a negative value)
- The right-hand side is a variable expression (such as x or $x - 2$) that could be negative for some values
- You square a binomial on either side and produce a quadratic with two solutions

Pro tip

- Before solving, identify the domain restriction. Write it down. It takes ten seconds and it tells you which answers to be suspicious of.
- After solving, if you have two solutions, always check both. The one that satisfies the original equation is valid. The one that does not is extraneous.

- Remember: a solution can be mathematically valid in the squared equation and still be extraneous in the original. Algebra alone is not enough — you must check.
- Cube root equations do not require domain checking, but it is still good practice to verify your answer.

Examples

Example A

Find the domain of $\sqrt{4x-12}$.

Step 1: Set the radicand greater than or equal to zero.

$$4x - 12 \geq 0$$

Step 2: Solve the inequality.

$$4x \geq 12$$

$$x \geq 3$$

The domain is all real numbers $x \geq 3$.

Example B

Find the domain of $\sqrt{10-2x}$.

Step 1: Set the radicand greater than or equal to zero.

$$10 - 2x \geq 0$$

Step 2: Solve the inequality. Subtract 10 from both sides.

$$-2x \geq -10$$

Step 3: Divide both sides by -2. Remember: when dividing an inequality by a negative number, the inequality sign flips.

$$x \leq 5$$

The domain is all real numbers $x \leq 5$.

Example C

Solve $\sqrt{x+8} = x + 2$ and identify any extraneous roots.

Step 1: Find the domain restriction.

$$x + 8 \geq 0 \qquad \Rightarrow \qquad x \geq -8$$

Also, since $\sqrt{x+8} \geq 0$, we need $x + 2 \geq 0$, so $x \geq -2$.
Step 2: Square both sides.

$$x + 8 = (x+2)^2 = x^2 + 4x + 4$$

Step 3: Rearrange into standard form.

$$0 = x^2 + 4x + 4 - x - 8$$

$$0 = x^2 + 3x - 4$$

Step 4: Factor.

$$(x + 4)(x - 1) = 0$$

Step 5: Solve.

$$x = -4 \qquad \text{or} \qquad x = 1$$

Step 6: Note that $x = -4$ already violates our domain restriction of $x \geq -2$. It is likely extraneous.

Step 7: Check $x = 1$.

$$\sqrt{1 + 8} = \sqrt{9} = 3 \qquad \text{and} \qquad 1 + 2 = 3\checkmark$$

Step 8: Check $x = -4$.

$$\sqrt{-4 + 8} = \sqrt{4} = 2 \qquad \text{and} \qquad -4 + 2 = -2$$

$$2 \neq -2$$

$x = -4$ is extraneous. The solution is $x = 1$.

Example D

Solve $\sqrt[3]{2x - 1} = 3$.

Step 1: There is no domain restriction for a cube root.

Step 2: Cube both sides to eliminate the cube root.

$$2x - 1 = 3^3 = 27$$

Step 3: Solve.

$$2x = 28 \qquad \Rightarrow \qquad x = 14$$

Step 4: Check.

$$\sqrt[3]{2(14) - 1} = \sqrt[3]{27} = 3\checkmark$$

The solution is $x = 14$.

Example E

State the domain of $\dfrac{1}{\sqrt{x - 4}}$.

Step 1: There are two restrictions here. The expression under the square root cannot be negative, and the denominator cannot equal zero.

Step 2: Set the radicand strictly greater than zero (not just ≥ 0, because the denominator cannot be zero).

$$x - 4 > 0$$

Step 3: Solve.

$$x > 4$$

The domain is all real numbers $x > 4$.

Common mistakes

- **Forgetting to flip the inequality sign when dividing by a negative.** When you solve $-2x \geq -10$, dividing by -2 gives $x \leq 5$, not $x \geq 5$.
- **Assuming a solution is valid just because it satisfies the domain.** Being in the domain is necessary but not sufficient. You must still check the solution in the original equation.
- **Confusing domain restriction with extraneous roots.** A value outside the domain will always be extraneous, but a value inside the domain can still be extraneous — it depends on whether it satisfies the original equation.
- **Applying square root domain rules to cube roots.** Cube roots have no domain restriction on the radicand. The expression $\sqrt[3]{-27}$ is perfectly valid and equals -3.

Worksheet

1. State the domain of each radical expression.

   (a) $\sqrt{x}$

   (b) $\sqrt{x-5}$

   (c) $\sqrt{x+9}$

   (d) $\sqrt{3x-15}$

2. State the domain of each radical expression.

   (a) $\sqrt{8-x}$

   (b) $\sqrt{12-4x}$

(c) $\sqrt{6-3x}$

(d) $\sqrt{20-5x}$

3. State the domain of each expression. Note any additional restrictions.

(a) $\dfrac{1}{\sqrt{x+2}}$

(b) $\dfrac{5}{\sqrt{3x-9}}$

(c) $\sqrt[3]{x+7}$

4. Solve each equation and identify any extraneous roots.

(a) $\sqrt{x-2} = x-4$

(b) $\sqrt{x+6} = x$

5. Solve each equation. Check for extraneous roots.

(a) $\sqrt{2x+5} = x+1$

(b) $\sqrt{x+12} = x$

6. Solve $\sqrt[3]{3x+2} = 4$.

7. Solve $\sqrt[3]{5x-7} = -3$.

8. Explain in your own words why extraneous roots appear when solving radical equations. Your explanation should be at least three sentences.

9. **Challenge:** Find the domain of $\sqrt{x^2-9}$.

   Hint: You need $x^2 - 9 \geq 0$. Factor and use a sign analysis.

10. **Challenge:** Solve $\sqrt{x+3} + \sqrt{x-2} = 5$ and check for extraneous roots.

    Hint: Isolate one radical, square both sides, then isolate the remaining radical and square again.

Answer Key

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1.

(a) Domain of $\sqrt{x}$.
Step 1: Set the radicand ≥ 0.

$$x \geq 0$$

$$\boxed{x \geq 0}$$

(b) Domain of $\sqrt{x-5}$.
Step 1: Set the radicand ≥ 0.

$$x - 5 \geq 0$$

Step 2: Solve.

$$x \geq 5$$

$$\boxed{x \geq 5}$$

(c) Domain of $\sqrt{x+9}$.
Step 1: Set the radicand ≥ 0.

$$x + 9 \geq 0$$

Step 2: Solve.

$$x \geq -9$$

$$\boxed{x \geq -9}$$

(d) Domain of $\sqrt{3x-15}$.
Step 1: Set the radicand ≥ 0.

$$3x - 15 \geq 0$$

Step 2: Solve.

$$3x \geq 15 \qquad \Rightarrow \qquad x \geq 5$$

$$\boxed{x \geq 5}$$

2.

(a) Domain of $\sqrt{8-x}$.
Step 1: Set the radicand ≥ 0.

$$8 - x \geq 0$$

Step 2: Subtract 8 from both sides.

$$-x \geq -8$$

Step 3: Divide by -1 and flip the inequality sign.

$$x \leq 8$$

$$\boxed{x \leq 8}$$

(b) Domain of $\sqrt{12 - 4x}$.
Step 1: Set the radicand ≥ 0.

$$12 - 4x \geq 0$$

Step 2: Subtract 12 from both sides.

$$-4x \geq -12$$

Step 3: Divide by -4 and flip the inequality sign.

$$x \leq 3$$

$$\boxed{x \leq 3}$$

(c) Domain of $\sqrt{6 - 3x}$.
Step 1: Set the radicand ≥ 0.

$$6 - 3x \geq 0$$

Step 2: Subtract 6 from both sides.

$$-3x \geq -6$$

Step 3: Divide by -3 and flip the inequality sign.

$$x \leq 2$$

$$\boxed{x \leq 2}$$

(d) Domain of $\sqrt{20 - 5x}$.
Step 1: Set the radicand ≥ 0.

$$20 - 5x \geq 0$$

Step 2: Subtract 20 from both sides.

$$-5x \geq -20$$

Step 3: Divide by -5 and flip the inequality sign.

$$x \leq 4$$

$$\boxed{x \leq 4}$$

3.

(a) Domain of $\dfrac{1}{\sqrt{x+2}}$.
Step 1: The radicand must be ≥ 0 and the denominator cannot equal zero.
Step 2: Set the radicand strictly greater than zero.

$$x + 2 > 0 \qquad \Rightarrow \qquad x > -2$$

$$\boxed{x > -2}$$

(b) Domain of $\dfrac{5}{\sqrt{3x-9}}$.
Step 1: Set the radicand strictly greater than zero (denominator cannot be zero).

$$3x - 9 > 0$$

Step 2: Solve.

$$3x > 9 \qquad \Rightarrow \qquad x > 3$$

$$\boxed{x > 3}$$

(c) Domain of $\sqrt[3]{x+7}$.
Step 1: Cube roots are defined for all real numbers. There is no restriction on the radicand.

$$\boxed{\text{All real numbers}}$$

4.

(a) Solve $\sqrt{x-2} = x - 4$.
Step 1: Note the domain restriction: $x - 2 \geq 0$ so $x \geq 2$. Also the right side must be ≥ 0, so $x \geq 4$.
Step 2: Square both sides.

$$x - 2 = (x-4)^2 = x^2 - 8x + 16$$

Step 3: Rearrange.

$$0 = x^2 - 8x + 16 - x + 2$$
$$0 = x^2 - 9x + 18$$

Step 4: Factor.

$$(x-6)(x-3) = 0$$

Step 5: Solve.

$$x = 6 \qquad \text{or} \qquad x = 3$$

Step 6: Check $x = 6$.

$$\sqrt{6-2} = \sqrt{4} = 2 \qquad \text{and} \qquad 6 - 4 = 2\checkmark$$

Step 7: Check $x = 3$.

$$\sqrt{3-2} = \sqrt{1} = 1 \qquad \text{and} \qquad 3 - 4 = -1$$

$$1 \neq -1$$

$x = 3$ is extraneous. Note it also violates the domain restriction $x \geq 4$.

$$\boxed{x = 6}$$

(b) Solve $\sqrt{x+6} = x$.
Step 1: Domain: $x + 6 \geq 0$ so $x \geq -6$. Also $x \geq 0$ since the left side is non-negative.
Step 2: Square both sides.

$$x + 6 = x^2$$

Step 3: Rearrange.

$$x^2 - x - 6 = 0$$

Step 4: Factor.

$$(x-3)(x+2) = 0$$

Step 5: Solve.

$$x = 3 \qquad \text{or} \qquad x = -2$$

Step 6: Check $x = 3$.

$$\sqrt{3+6} = \sqrt{9} = 3\checkmark$$

Step 7: Check $x = -2$.

$$\sqrt{-2+6} = \sqrt{4} = 2 \neq -2$$

$x = -2$ is extraneous.

$$\boxed{x = 3}$$

5.

(a) Solve $\sqrt{2x+5} = x + 1$.
Step 1: Domain: $2x + 5 \geq 0$ so $x \geq -\frac{5}{2}$. Also $x + 1 \geq 0$ so $x \geq -1$.
Step 2: Square both sides.

$$2x + 5 = (x+1)^2 = x^2 + 2x + 1$$

Step 3: Rearrange.

$$0 = x^2 + 2x + 1 - 2x - 5$$
$$0 = x^2 - 4$$

Step 4: Solve.

$$x^2 = 4 \qquad \Rightarrow \qquad x = 2 \quad \text{or} \quad x = -2$$

Step 5: Check $x = 2$.

$$\sqrt{2(2)+5} = \sqrt{9} = 3 \qquad \text{and} \qquad 2 + 1 = 3\checkmark$$

Step 6: Check $x = -2$.

$$\sqrt{2(-2)+5} = \sqrt{1} = 1 \qquad \text{and} \qquad -2+1 = -1$$

$$1 \neq -1$$

$x = -2$ is extraneous.

$$\boxed{x = 2}$$

(b) Solve $\sqrt{x+12} = x$.
Step 1: Domain: $x + 12 \geq 0$ so $x \geq -12$. Also $x \geq 0$.
Step 2: Square both sides.

$$x + 12 = x^2$$

Step 3: Rearrange.

$$x^2 - x - 12 = 0$$

Step 4: Factor.

$$(x-4)(x+3) = 0$$

Step 5: Solve.

$$x = 4 \qquad \text{or} \qquad x = -3$$

Step 6: Check $x = 4$.

$$\sqrt{4+12} = \sqrt{16} = 4\checkmark$$

Step 7: Check $x = -3$.

$$\sqrt{-3+12} = \sqrt{9} = 3 \neq -3$$

$x = -3$ is extraneous.

$$\boxed{x = 4}$$

6. Solve $\sqrt[3]{3x+2} = 4$.
Step 1: No domain restriction for a cube root.
Step 2: Cube both sides.

$$3x + 2 = 4^3 = 64$$

Step 3: Solve.

$$3x = 62 \qquad \Rightarrow \qquad x = \frac{62}{3}$$

Step 4: Check.

$$\sqrt[3]{3 \times \frac{62}{3} + 2} = \sqrt[3]{62+2} = \sqrt[3]{64} = 4\checkmark$$

$$\boxed{x = \frac{62}{3}}$$

7. Solve $\sqrt[3]{5x-7} = -3$.

Step 1: No domain restriction for a cube root. Note that unlike square roots, cube roots **can** equal negative numbers.

Step 2: Cube both sides.

$$5x - 7 = (-3)^3 = -27$$

Step 3: Solve.

$$5x = -27 + 7 = -20 \qquad \Rightarrow \qquad x = -4$$

Step 4: Check.

$$\sqrt[3]{5(-4) - 7} = \sqrt[3]{-20 - 7} = \sqrt[3]{-27} = -3\checkmark$$

$$\boxed{x = -4}$$

8. Explanation of why extraneous roots appear.

A full explanation should include the following ideas:

When we solve a radical equation, we square both sides to remove the square root. Squaring is not a reversible operation — it loses information about sign. Both 3 and -3 give 9 when squared, so squaring cannot distinguish between positive and negative values. This means that squaring can introduce solutions that satisfy the squared equation but not the original equation, where the square root was present. Because a square root always produces a non-negative result, any solution that would require the square root to equal a negative number is extraneous and must be rejected. Checking every solution in the original equation is the only reliable way to identify and discard these false solutions.

9. Find the domain of $\sqrt{x^2 - 9}$.

Step 1: Set the radicand ≥ 0.

$$x^2 - 9 \geq 0$$

Step 2: Factor.

$$(x - 3)(x + 3) \geq 0$$

Step 3: Identify the critical values. The expression equals zero at $x = 3$ and $x = -3$.

Step 4: Test each region using a sign chart.

- For $x < -3$, test $x = -4$: $(-4 - 3)(-4 + 3) = (-7)(-1) = 7 > 0$ ✓
- For $-3 < x < 3$, test $x = 0$: $(0 - 3)(0 + 3) = (-3)(3) = -9 < 0$ ×
- For $x > 3$, test $x = 4$: $(4 - 3)(4 + 3) = (1)(7) = 7 > 0$ ✓

Step 5: Include the endpoints since the inequality allows equality.

$$\boxed{x \leq -3 \quad \text{or} \quad x \geq 3}$$

10. Solve $\sqrt{x + 3} + \sqrt{x - 2} = 5$.

Step 1: Domain requires $x + 3 \geq 0$ and $x - 2 \geq 0$, so $x \geq 2$.

Step 2: Isolate one radical. Move $\sqrt{x-2}$ to the right side.

$$\sqrt{x+3} = 5 - \sqrt{x-2}$$

Step 3: Square both sides.

$$x + 3 = (5 - \sqrt{x-2})^2 = 25 - 10\sqrt{x-2} + (x-2)$$

$$x + 3 = 23 + x - 10\sqrt{x-2}$$

Step 4: Simplify. Subtract x from both sides.

$$3 = 23 - 10\sqrt{x-2}$$

Step 5: Isolate the remaining radical.

$$10\sqrt{x-2} = 23 - 3 = 20$$

$$\sqrt{x-2} = 2$$

Step 6: Square both sides.

$$x - 2 = 4 \qquad \Rightarrow \qquad x = 6$$

Step 7: Check.

$$\sqrt{6+3} + \sqrt{6-2} = \sqrt{9} + \sqrt{4} = 3 + 2 = 5\checkmark$$

$$\boxed{x = 6}$$

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