Adding and Subtracting Rational Expressions
ApexMath

Notes

The Big Idea

Adding and subtracting rational expressions works exactly like adding and subtracting numerical fractions. The key rule is:

You can only add or subtract fractions that have the same denominator.

Recall with numbers:

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4} \qquad \text{(same denominator — add numerators directly)}$$

$$\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12} \qquad \text{(different denominators — find a common one first)}$$

Rational expressions follow the exact same process.

Case 1 — Same Denominator

When rational expressions already share a denominator, simply add or subtract the numerators and keep the denominator.

$$\frac{A}{D} + \frac{B}{D} = \frac{A+B}{D} \qquad \frac{A}{D} - \frac{B}{D} = \frac{A-B}{D}$$

After combining, always simplify the result fully and state NPVs.

Case 2 — Different Denominators

When the denominators are different, you must find the **Lowest Common Denominator** (LCD) before combining.

How to find the LCD:

- Step 1: Fully factor each denominator.
- Step 2: The LCD is the product of all unique factors, each taken to the highest power it appears in any single denominator.

For example, if the denominators are $(x+2)$ and $(x-3)$, the LCD is $(x+2)(x-3)$.

If the denominators are $(x)(x+1)$ and $(x+1)$, the LCD is $x(x+1)$ — not $x(x+1)^2$, because $(x+1)$ only needs to appear once.

The Full Method

- Step 1: Factor all denominators completely.
- Step 2: State all non-permissible values.
- Step 3: Identify the LCD.

- Step 4: Rewrite each fraction with the LCD as its denominator by multiplying the numerator and denominator by the missing factor(s).
- Step 5: Add or subtract the numerators. Keep the LCD as the denominator.
- Step 6: Expand and simplify the numerator.
- Step 7: Factor the numerator if possible and cancel any common factors with the denominator.

Watch the Subtraction Sign

When subtracting rational expressions, the subtraction sign applies to the *entire* numerator of the second fraction, not just the first term. Use brackets to keep track.

$$\frac{A}{D} - \frac{B+C}{D} = \frac{A-(B+C)}{D} = \frac{A-B-C}{D}$$

Forgetting to distribute the negative sign is one of the most common errors in this topic.

Pro tip

- Always factor denominators first before looking for the LCD. You cannot identify the LCD from unfactored expressions.
- When you rewrite a fraction with the LCD, multiply both the numerator **and** denominator by the missing factor. Only changing the denominator changes the value of the fraction.
- Put brackets around the numerator you are subtracting. Distribute the negative sign through every term inside the brackets.
- After combining, always try to factor the numerator and check whether it shares a common factor with the denominator. If it does, cancel and simplify.
- Double-check your NPVs come from the original factored denominators, not the simplified final expression.

Examples

Example A — Same Denominator, Addition

Simplify $\frac{3x}{x+5} + \frac{2x-1}{x+5}$. State all NPVs.

Step 1: The denominators are the same. NPV: $x \neq -5$.

Step 2: Add the numerators.

$$\frac{3x+(2x-1)}{x+5} = \frac{5x-1}{x+5}$$

Step 3: Check whether the numerator factors and cancels with the denominator. $5x - 1$ does not factor to match $(x + 5)$, so this is fully simplified.

The answer is $\dfrac{5x-1}{x+5}$, where $x \neq -5$.

Example B — Same Denominator, Subtraction

Simplify $\dfrac{4x+3}{x-2} - \dfrac{x+7}{x-2}$. State all NPVs.

Step 1: The denominators are the same. NPV: $x \neq 2$.

Step 2: Subtract the numerators. Use brackets around the second numerator.

$$\frac{(4x+3)-(x+7)}{x-2}$$

Step 3: Distribute the negative sign.

$$\frac{4x+3-x-7}{x-2} = \frac{3x-4}{x-2}$$

Step 4: $3x - 4$ does not share a factor with $(x - 2)$, so this is the final answer.

The answer is $\dfrac{3x-4}{x-2}$, where $x \neq 2$.

Example C — Different Denominators, Simple

Simplify $\dfrac{3}{x} + \dfrac{5}{x+2}$. State all NPVs.

Step 1: Denominators are x and $(x + 2)$. Both are already fully factored.

Step 2: NPVs: $x \neq 0$ and $x \neq -2$.

Step 3: LCD $= x(x + 2)$.

Step 4: Rewrite each fraction with the LCD.

$$\frac{3}{x} \times \frac{(x+2)}{(x+2)} = \frac{3(x+2)}{x(x+2)}$$

$$\frac{5}{x+2} \times \frac{x}{x} = \frac{5x}{x(x+2)}$$

Step 5: Add the numerators.

$$\frac{3(x+2)+5x}{x(x+2)}$$

Step 6: Expand and simplify the numerator.

$$3x + 6 + 5x = 8x + 6$$

Step 7: The numerator is $8x + 6 = 2(4x + 3)$. Check whether $(4x + 3)$ shares a factor with $x(x + 2)$. It does not.

The answer is $\dfrac{2(4x+3)}{x(x+2)}$, where $x \neq 0$ and $x \neq -2$.

Example D — Different Denominators, Factoring Required

Simplify $\dfrac{2}{x+3} - \dfrac{1}{x^2-9}$. State all NPVs.

Step 1: Factor all denominators.

$$x^2 - 9 = (x-3)(x+3)$$

Step 2: NPVs: $x \neq -3$ and $x \neq 3$.

Step 3: LCD $= (x-3)(x+3)$.

Step 4: The first fraction is missing the factor $(x-3)$. Multiply its numerator and denominator by $(x-3)$.

$$\frac{2}{x+3} \times \frac{(x-3)}{(x-3)} = \frac{2(x-3)}{(x-3)(x+3)}$$

The second fraction already has the LCD as its denominator.

$$\frac{1}{(x-3)(x+3)}$$

Step 5: Subtract.

$$\frac{2(x-3)-1}{(x-3)(x+3)}$$

Step 6: Expand the numerator.

$$2x - 6 - 1 = 2x - 7$$

Step 7: $2x-7$ does not factor to match any part of the denominator.

The answer is $\dfrac{2x-7}{(x-3)(x+3)}$, where $x \neq 3$ and $x \neq -3$.

Example E — Result Simplifies Further

Simplify $\dfrac{x}{x-4} + \dfrac{-4}{x-4}$. State all NPVs.

Step 1: The denominators are the same. NPV: $x \neq 4$.

Step 2: Add the numerators.

$$\frac{x+(-4)}{x-4} = \frac{x-4}{x-4}$$

Step 3: Cancel the common factor $(x-4)$.

$$\frac{x-4}{x-4} = 1$$

The answer is 1, where $x \neq 4$.

Common mistakes

- **Not distributing the negative sign when subtracting.** $\frac{A}{D} - \frac{B+C}{D} = \frac{A-B-C}{D}$, not $\frac{A-B+C}{D}$. Always put brackets around the subtracted numerator.
- **Only changing the denominator when converting to the LCD.** When you multiply the denominator by a missing factor, you must multiply the numerator by the same factor too.
- **Adding denominators.** $\frac{1}{x} + \frac{1}{x+1} \neq \frac{2}{x(x+1)}$. You must rewrite both fractions with the LCD first.
- **Using too large an LCD.** If the denominators share a common factor, the LCD uses that factor only once. For example, if both denominators have $(x+2)$, it appears only once in the LCD.
- **Forgetting to simplify the final result.** After combining, always factor the numerator and check for cancellation with the denominator.

Worksheet

1. Add or subtract. The denominators are already the same. State all NPVs.

   (a) $\frac{2x}{x+1} + \frac{3}{x+1}$

   (b) $\frac{5x+2}{x-3} - \frac{2x-1}{x-3}$

   (c) $\frac{x^2}{x+6} - \frac{36}{x+6}$

   (d) $\frac{3x+4}{2x-1} + \frac{x-4}{2x-1}$

2. Add or subtract. Find the LCD first. State all NPVs.

(a) $\dfrac{2}{x} + \dfrac{3}{x+1}$

(b) $\dfrac{4}{x-2} - \dfrac{1}{x+3}$

(c) $\dfrac{x}{x+4} + \dfrac{2}{x-1}$

(d) $\dfrac{3}{x-5} - \dfrac{2}{x}$

3. Factor the denominators, find the LCD, then add or subtract. State all NPVs.

(a) $\dfrac{3}{x^2-4} + \dfrac{1}{x+2}$

(b) $\dfrac{5}{x^2-9} - \dfrac{2}{x+3}$

(c) $\dfrac{x}{x^2+5x+6} + \dfrac{1}{x+2}$

4. Simplify each expression. State all NPVs.

   (a) $\dfrac{2}{x+1} + \dfrac{x}{x^2-1}$

   (b) $\dfrac{x}{x-3} - \dfrac{6}{x^2-9}$

5. Simplify $\dfrac{3x}{x^2-x-6} - \dfrac{2}{x-3}$. State all NPVs.

6. Simplify $\dfrac{x+1}{x^2-4} + \dfrac{x-1}{x+2}$. State all NPVs.

7. A student simplified $\dfrac{3}{x-2} - \dfrac{1}{x+2}$ and got $\dfrac{2}{x^2-4}$. Find and correct the mistake.

8. **Challenge:** Simplify $\dfrac{2x}{x^2+3x+2} - \dfrac{1}{x+1} + \dfrac{3}{x+2}$. State all NPVs.

Answer Key

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1.

(a) $\dfrac{2x}{x+1} + \dfrac{3}{x+1}$

Step 1: Same denominator. NPV: $x \neq -1$.

Step 2: Add numerators.

$$\frac{2x+3}{x+1}$$

Step 3: $2x+3$ does not factor to match $(x+1)$.

$$\boxed{\frac{2x+3}{x+1}, \quad x \neq -1}$$

(b) $\dfrac{5x+2}{x-3} - \dfrac{2x-1}{x-3}$

Step 1: Same denominator. NPV: $x \neq 3$.

Step 2: Subtract numerators. Use brackets.

$$\frac{(5x+2)-(2x-1)}{x-3} = \frac{5x+2-2x+1}{x-3} = \frac{3x+3}{x-3}$$

Step 3: Factor the numerator. $3x+3 = 3(x+1)$. No cancellation with $(x-3)$.

$$\boxed{\frac{3(x+1)}{x-3}, \quad x \neq 3}$$

(c) $\dfrac{x^2}{x+6} - \dfrac{36}{x+6}$

Step 1: Same denominator. NPV: $x \neq -6$.

Step 2: Subtract numerators.

$$\frac{x^2-36}{x+6}$$

Step 3: Factor the numerator.

$$x^2 - 36 = (x-6)(x+6)$$

Step 4: Cancel $(x+6)$.

$$\frac{(x-6)(x+6)}{(x+6)} = x - 6$$

$$\boxed{x-6, \quad x \neq -6}$$

(d) $\dfrac{3x+4}{2x-1} + \dfrac{x-4}{2x-1}$

Step 1: Same denominator. NPV: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$, so $x \neq \frac{1}{2}$.
Step 2: Add numerators.

$$\frac{(3x+4)+(x-4)}{2x-1} = \frac{4x}{2x-1}$$

Step 3: $4x$ and $2x - 1$ share no common factors.

$$\boxed{\frac{4x}{2x-1}, \quad x \neq \frac{1}{2}}$$

2.

(a) $\frac{2}{x} + \frac{3}{x+1}$
Step 1: NPVs: $x \neq 0$ and $x \neq -1$.
Step 2: LCD $= x(x+1)$.
Step 3: Rewrite each fraction.

$$\frac{2(x+1)}{x(x+1)} + \frac{3x}{x(x+1)}$$

Step 4: Add numerators.

$$\frac{2(x+1)+3x}{x(x+1)} = \frac{2x+2+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$$

$$\boxed{\frac{5x+2}{x(x+1)}, \quad x \neq 0,\ x \neq -1}$$

(b) $\frac{4}{x-2} - \frac{1}{x+3}$
Step 1: NPVs: $x \neq 2$ and $x \neq -3$.
Step 2: LCD $= (x-2)(x+3)$.
Step 3: Rewrite each fraction.

$$\frac{4(x+3)}{(x-2)(x+3)} - \frac{1(x-2)}{(x-2)(x+3)}$$

Step 4: Subtract numerators. Use brackets.

$$\frac{4(x+3)-(x-2)}{(x-2)(x+3)} = \frac{4x+12-x+2}{(x-2)(x+3)} = \frac{3x+14}{(x-2)(x+3)}$$

$$\boxed{\frac{3x+14}{(x-2)(x+3)}, \quad x \neq 2,\ x \neq -3}$$

(c) $\dfrac{x}{x+4} + \dfrac{2}{x-1}$
Step 1: NPVs: $x \neq -4$ and $x \neq 1$.
Step 2: LCD $= (x+4)(x-1)$.
Step 3: Rewrite each fraction.

$$\frac{x(x-1)}{(x+4)(x-1)} + \frac{2(x+4)}{(x+4)(x-1)}$$

Step 4: Add numerators.

$$\frac{x(x-1)+2(x+4)}{(x+4)(x-1)} = \frac{x^2-x+2x+8}{(x+4)(x-1)} = \frac{x^2+x+8}{(x+4)(x-1)}$$

Step 5: x^2+x+8 does not factor over the integers.

$$\boxed{\frac{x^2+x+8}{(x+4)(x-1)}, \quad x \neq -4,\ x \neq 1}$$

(d) $\dfrac{3}{x-5} - \dfrac{2}{x}$
Step 1: NPVs: $x \neq 5$ and $x \neq 0$.
Step 2: LCD $= x(x-5)$.
Step 3: Rewrite each fraction.

$$\frac{3x}{x(x-5)} - \frac{2(x-5)}{x(x-5)}$$

Step 4: Subtract numerators.

$$\frac{3x-2(x-5)}{x(x-5)} = \frac{3x-2x+10}{x(x-5)} = \frac{x+10}{x(x-5)}$$

$$\boxed{\frac{x+10}{x(x-5)}, \quad x \neq 0,\ x \neq 5}$$

3.

(a) $\dfrac{3}{x^2-4} + \dfrac{1}{x+2}$
Step 1: Factor. $x^2-4 = (x-2)(x+2)$.
Step 2: NPVs: $x \neq 2$ and $x \neq -2$.
Step 3: LCD $= (x-2)(x+2)$.
Step 4: First fraction already has LCD. Rewrite the second.

$$\frac{3}{(x-2)(x+2)} + \frac{1(x-2)}{(x-2)(x+2)}$$

Step 5: Add numerators.

$$\frac{3+(x-2)}{(x-2)(x+2)} = \frac{x+1}{(x-2)(x+2)}$$

Step 6: $x+1$ shares no factor with the denominator.

$$\boxed{\frac{x+1}{(x-2)(x+2)}, \quad x \neq 2,\ x \neq -2}$$

(b) $\dfrac{5}{x^2-9} - \dfrac{2}{x+3}$
Step 1: Factor. $x^2 - 9 = (x-3)(x+3)$.
Step 2: NPVs: $x \neq 3$ and $x \neq -3$.
Step 3: LCD $= (x-3)(x+3)$.
Step 4: First fraction already has LCD. Rewrite the second.

$$\frac{5}{(x-3)(x+3)} - \frac{2(x-3)}{(x-3)(x+3)}$$

Step 5: Subtract numerators.

$$\frac{5-2(x-3)}{(x-3)(x+3)} = \frac{5-2x+6}{(x-3)(x+3)} = \frac{11-2x}{(x-3)(x+3)}$$

$$\boxed{\frac{11-2x}{(x-3)(x+3)}, \quad x \neq 3,\ x \neq -3}$$

(c) $\dfrac{x}{x^2+5x+6} + \dfrac{1}{x+2}$
Step 1: Factor. $x^2 + 5x + 6 = (x+2)(x+3)$.
Step 2: NPVs: $x \neq -2$ and $x \neq -3$.
Step 3: LCD $= (x+2)(x+3)$.
Step 4: First fraction already has LCD. Rewrite the second.

$$\frac{x}{(x+2)(x+3)} + \frac{1(x+3)}{(x+2)(x+3)}$$

Step 5: Add numerators.

$$\frac{x+(x+3)}{(x+2)(x+3)} = \frac{2x+3}{(x+2)(x+3)}$$

Step 6: $2x+3$ does not factor to match any denominator factor.

$$\boxed{\frac{2x+3}{(x+2)(x+3)}, \quad x \neq -2,\ x \neq -3}$$

4.

(a) $\dfrac{2}{x+1} + \dfrac{x}{x^2-1}$

Step 1: Factor. $x^2 - 1 = (x-1)(x+1)$.
Step 2: NPVs: $x \neq -1$ and $x \neq 1$.
Step 3: LCD $= (x-1)(x+1)$.
Step 4: Rewrite the first fraction.

$$\frac{2(x-1)}{(x-1)(x+1)} + \frac{x}{(x-1)(x+1)}$$

Step 5: Add numerators.

$$\frac{2(x-1)+x}{(x-1)(x+1)} = \frac{2x-2+x}{(x-1)(x+1)} = \frac{3x-2}{(x-1)(x+1)}$$

Step 6: $3x - 2$ does not factor to match the denominator.

$$\boxed{\frac{3x-2}{(x-1)(x+1)}, \quad x \neq 1,\ x \neq -1}$$

(b) $\dfrac{x}{x-3} - \dfrac{6}{x^2-9}$

Step 1: Factor. $x^2 - 9 = (x-3)(x+3)$.
Step 2: NPVs: $x \neq 3$ and $x \neq -3$.
Step 3: LCD $= (x-3)(x+3)$.
Step 4: Rewrite the first fraction.

$$\frac{x(x+3)}{(x-3)(x+3)} - \frac{6}{(x-3)(x+3)}$$

Step 5: Subtract numerators.

$$\frac{x(x+3)-6}{(x-3)(x+3)} = \frac{x^2+3x-6}{(x-3)(x+3)}$$

Step 6: $x^2 + 3x - 6$ does not factor over the integers.

$$\boxed{\frac{x^2+3x-6}{(x-3)(x+3)}, \quad x \neq 3,\ x \neq -3}$$

5. Simplify $\dfrac{3x}{x^2-x-6} - \dfrac{2}{x-3}$.

Step 1: Factor. $x^2 - x - 6 = (x-3)(x+2)$.
Step 2: NPVs: $x \neq 3$ and $x \neq -2$.
Step 3: LCD $= (x-3)(x+2)$.

Step 4: First fraction already has LCD. Rewrite the second.

$$\frac{3x}{(x-3)(x+2)} - \frac{2(x+2)}{(x-3)(x+2)}$$

Step 5: Subtract numerators.

$$\frac{3x-2(x+2)}{(x-3)(x+2)} = \frac{3x-2x-4}{(x-3)(x+2)} = \frac{x-4}{(x-3)(x+2)}$$

Step 6: $x-4$ shares no factor with the denominator.

$$\boxed{\frac{x-4}{(x-3)(x+2)}, \quad x \neq 3,\ x \neq -2}$$

6. Simplify $\dfrac{x+1}{x^2-4} + \dfrac{x-1}{x+2}$.
Step 1: Factor. $x^2-4 = (x-2)(x+2)$.
Step 2: NPVs: $x \neq 2$ and $x \neq -2$.
Step 3: LCD $= (x-2)(x+2)$.
Step 4: First fraction already has LCD. Rewrite the second.

$$\frac{x+1}{(x-2)(x+2)} + \frac{(x-1)(x-2)}{(x-2)(x+2)}$$

Step 5: Add numerators.

$$\frac{(x+1)+(x-1)(x-2)}{(x-2)(x+2)}$$

Step 6: Expand $(x-1)(x-2) = x^2-3x+2$.

$$\frac{x+1+x^2-3x+2}{(x-2)(x+2)} = \frac{x^2-2x+3}{(x-2)(x+2)}$$

Step 7: x^2-2x+3 does not factor over the integers.

$$\boxed{\frac{x^2-2x+3}{(x-2)(x+2)}, \quad x \neq 2,\ x \neq -2}$$

7. A student simplified $\dfrac{3}{x-2} - \dfrac{1}{x+2}$ and got $\dfrac{2}{x^2-4}$.
Step 1: The student subtracted the numerators $(3-1=2)$ but also multiplied the denominators instead of finding the LCD correctly.
Step 2: Find the correct LCD.

$$\text{LCD} = (x-2)(x+2)$$

Step 3: Rewrite each fraction.

$$\frac{3(x+2)}{(x-2)(x+2)} - \frac{1(x-2)}{(x-2)(x+2)}$$

Step 4: Subtract numerators carefully.

$$\frac{3(x+2)-(x-2)}{(x-2)(x+2)} = \frac{3x+6-x+2}{(x-2)(x+2)} = \frac{2x+8}{(x-2)(x+2)}$$

Step 5: Factor the numerator.

$$2x+8 = 2(x+4)$$

Step 6: $x+4$ shares no factor with the denominator.

Step 7: State the student's error. The student correctly used $(x-2)(x+2)$ as the denominator but forgot to multiply each numerator by the missing factor before subtracting.

$$\boxed{\frac{2(x+4)}{(x-2)(x+2)}, \quad x \neq 2,\ x \neq -2}$$

8. Simplify $\frac{2x}{x^2+3x+2} - \frac{1}{x+1} + \frac{3}{x+2}$.

Step 1: Factor. $x^2+3x+2 = (x+1)(x+2)$.
Step 2: NPVs: $x \neq -1$ and $x \neq -2$.
Step 3: LCD $= (x+1)(x+2)$.
Step 4: Rewrite each fraction with the LCD.

$$\frac{2x}{(x+1)(x+2)} - \frac{1(x+2)}{(x+1)(x+2)} + \frac{3(x+1)}{(x+1)(x+2)}$$

Step 5: Combine all numerators in one fraction.

$$\frac{2x-(x+2)+3(x+1)}{(x+1)(x+2)}$$

Step 6: Expand and simplify. Distribute carefully.

$$2x - x - 2 + 3x + 3 = 4x + 1$$

Step 7: $4x+1$ does not factor to match any denominator factor.

$$\boxed{\frac{4x+1}{(x+1)(x+2)}, \quad x \neq -1,\ x \neq -2}$$

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