Solving Rational Equations
ApexMath

Notes

The Big Idea

A **rational equation** is an equation that contains at least one rational expression — that is, at least one fraction with a variable in the denominator.

Examples:

$$\frac{3}{x} = 6 \qquad \frac{x}{x-2} + 1 = \frac{4}{x-2} \qquad \frac{2}{x+1} = \frac{3}{x-1}$$

The strategy for solving these is different from the strategy for simplifying. When **simplifying**, you keep the denominator and work with fractions. When **solving an equation**, you can **eliminate the denominators entirely** by multiplying both sides by the LCD.

The Method — Multiplying Through by the LCD

- Step 1: Factor all denominators and identify all non-permissible values. Write them down before solving.
- Step 2: Find the LCD of all fractions in the equation.
- Step 3: Multiply every term on both sides of the equation by the LCD.
- Step 4: Simplify. Each fraction's denominator will cancel with part of the LCD, leaving a polynomial equation.
- Step 5: Solve the resulting equation (linear or quadratic).
- Step 6: Check each solution against the non-permissible values. Reject any solution that is an NPV.
- Step 7: Verify valid solutions by substituting back into the original equation.

Why NPVs Matter More Here Than Ever

When solving, you might get a solution that equals one of the NPVs. That solution must be rejected — it would make a denominator zero in the original equation, which is undefined. These are called **extraneous solutions**.

This is why you must identify NPVs at the very beginning, before any algebra takes place.

Proportions — A Special Case

When the equation has one fraction on each side and nothing else, it is a **proportion**:

$$\frac{A}{B} = \frac{C}{D}$$

You can solve a proportion by **cross-multiplying**:

$$A \times D = B \times C$$

This is a shortcut for multiplying both sides by the LCD BD.

Equations That Produce Quadratics

Sometimes after eliminating denominators, you will get a quadratic equation. In that case:

- Rearrange into standard form $ax^2 + bx + c = 0$.
- Factor and solve, or use the quadratic formula.
- Check both solutions against the NPVs and in the original equation.

Pro tip

- Write down your NPVs **before** you start solving. If you wait until the end, you might forget them.
- When you multiply through by the LCD, apply it to **every single term** on both sides — including whole number terms that have no denominator.
- Always verify your answer in the original equation, not in a simplified version of it. An extraneous solution can appear valid in a simplified equation but will fail in the original.
- If the equation produces two solutions from a quadratic, both must be checked separately.

Examples

Example A — Simple Rational Equation

Solve $\dfrac{3}{x} = 6$.

Step 1: NPV: $x \neq 0$.

Step 2: LCD $= x$.

Step 3: Multiply both sides by x.

$$x \cdot \frac{3}{x} = 6 \cdot x$$

$$3 = 6x$$

Step 4: Solve.

$$x = \frac{3}{6} = \frac{1}{2}$$

Step 5: Check NPV. $x = \frac{1}{2}$ is not zero, so it is valid.
Step 6: Verify. $\frac{3}{\frac{1}{2}} = 3 \times 2 = 6$ ✓
The solution is $x = \frac{1}{2}$.

Example B — Variable on Both Sides
Solve $\frac{x}{x-2} = \frac{3}{x-2} + 1$.
Step 1: NPV: $x - 2 = 0 \Rightarrow x \neq 2$.
Step 2: LCD $= x - 2$.
Step 3: Multiply every term by $(x - 2)$.

$$(x-2) \cdot \frac{x}{x-2} = (x-2) \cdot \frac{3}{x-2} + (x-2) \cdot 1$$

$$x = 3 + (x - 2)$$

$$x = 3 + x - 2$$

$$x = x + 1$$

Step 4: This gives $0 = 1$, which is a contradiction. There is no solution.
The equation has **no solution**.

Example C — Proportion
Solve $\frac{x+1}{3} = \frac{2x-1}{5}$.
Step 1: No variables in denominators, so no NPVs.
Step 2: Cross-multiply.

$$5(x + 1) = 3(2x - 1)$$

Step 3: Expand both sides.

$$5x + 5 = 6x - 3$$

Step 4: Solve.

$$5 + 3 = 6x - 5x$$

$$x = 8$$

Step 5: Verify. $\frac{8+1}{3} = \frac{9}{3} = 3$ and $\frac{2(8)-1}{5} = \frac{15}{5} = 3$ ✓
The solution is $x = 8$.

Example D — Two Different Denominators
Solve $\frac{2}{x+1} + \frac{1}{x-1} = \frac{4}{x^2-1}$.
Step 1: Factor. $x^2 - 1 = (x - 1)(x + 1)$.
NPVs: $x \neq -1$ and $x \neq 1$.
Step 2: LCD $= (x + 1)(x - 1)$.

Step 3: Multiply every term by the LCD.

$$(x+1)(x-1)\cdot\frac{2}{x+1}+(x+1)(x-1)\cdot\frac{1}{x-1}=(x+1)(x-1)\cdot\frac{4}{(x-1)(x+1)}$$

Step 4: Cancel denominators.

$$2(x-1)+1(x+1)=4$$

Step 5: Expand and solve.

$$2x-2+x+1=4$$
$$3x-1=4$$
$$3x=5$$
$$x=\frac{5}{3}$$

Step 6: Check NPVs. $x=\frac{5}{3}$ is neither 1 nor -1. Valid.
Step 7: Verify in original equation.

$$\frac{2}{\frac{5}{3}+1}+\frac{1}{\frac{5}{3}-1}=\frac{2}{\frac{8}{3}}+\frac{1}{\frac{2}{3}}=\frac{6}{8}+\frac{3}{2}=\frac{3}{4}+\frac{6}{4}=\frac{9}{4}$$

$$\frac{4}{x^2-1}=\frac{4}{\frac{25}{9}-1}=\frac{4}{\frac{16}{9}}=\frac{36}{16}=\frac{9}{4}\checkmark$$

The solution is $x=\frac{5}{3}$.

Example E — Quadratic Result with an Extraneous Solution
Solve $\frac{x}{x-3}-\frac{3}{x+3}=\frac{18}{x^2-9}$.
Step 1: Factor. $x^2-9=(x-3)(x+3)$.
NPVs: $x\neq 3$ and $x\neq -3$.
Step 2: LCD $=(x-3)(x+3)$.
Step 3: Multiply every term by the LCD.

$$x(x+3)-3(x-3)=18$$

Step 4: Expand.

$$x^2+3x-3x+9=18$$
$$x^2+9=18$$

Step 5: Solve.

$$x^2=9$$
$$x=3 \quad \text{or} \quad x=-3$$

Step 6: Check NPVs. Both $x=3$ and $x=-3$ are NPVs. Both solutions are extraneous.
The equation has **no solution**.

Common mistakes

- **Forgetting to multiply every term by the LCD.** Every term on both sides must be multiplied — including whole number terms with no denominator.
- **Not checking solutions against NPVs.** A solution that equals an NPV must be rejected. It makes the original equation undefined.
- **Confusing solving with simplifying.** When simplifying, you keep the denominator. When solving, you multiply both sides by the LCD to eliminate denominators entirely.
- **Expanding a binomial incorrectly.** When the LCD involves a binomial, multiply it out carefully. For example, $2(x-1) = 2x-2$, not $2x-1$.
- **Stopping at the NPV check.** Even if a solution is not an NPV, you should still verify it in the original equation.

Worksheet

1. Solve each equation. State NPVs and verify your answer.

   (a) $\dfrac{6}{x} = 2$

   (b) $\dfrac{10}{x} = 5$

   (c) $\dfrac{4}{x+2} = 1$

   (d) $\dfrac{3}{2x-4} = 1$

2. Solve each proportion by cross-multiplying. Verify your answer.

(a) $\dfrac{x}{4} = \dfrac{3}{2}$

(b) $\dfrac{x+1}{5} = \dfrac{2x-1}{3}$

(c) $\dfrac{3}{x-1} = \dfrac{5}{x+2}$

3. Solve each equation. State NPVs. Check for extraneous solutions.

(a) $\dfrac{2}{x-3} + \dfrac{1}{x+3} = \dfrac{6}{x^2-9}$

(b) $\dfrac{3}{x+2} - \dfrac{1}{x-2} = \dfrac{8}{x^2-4}$

4. Solve each equation. Check for extraneous solutions.

(a) $\dfrac{x}{x-4} = \dfrac{4}{x-4} + 2$

(b) $\dfrac{x+3}{x} = \dfrac{4}{x} + 1$

5. Solve $\dfrac{x}{x+2} + \dfrac{2}{x-3} = \dfrac{10}{x^2-x-6}$. State all NPVs and check for extraneous solutions.

6. Solve $\dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{1}{4}$. State all NPVs and verify.

7. A student solved $\dfrac{2}{x-1} = \dfrac{3}{x+1}$ and got $x = 5$. Verify whether this is correct and show a full solution.

8. **Challenge:** Solve $\dfrac{x}{x-2} + \dfrac{x}{x+2} = \dfrac{8}{x^2-4}$. State all NPVs and check for extraneous solutions.

9. **Challenge:** Solve $\dfrac{3}{x+2} + \dfrac{2}{x-3} = \dfrac{5x-3}{x^2-x-6}$. State all NPVs.

Answer Key

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1.

(a) $\dfrac{6}{x} = 2$
Step 1: NPV: $x \neq 0$.
Step 2: Multiply both sides by x.

$$6 = 2x \quad \Rightarrow \quad x = 3$$

Step 3: $x = 3 \neq 0$. Valid.
Step 4: Verify: $\dfrac{6}{3} = 2$ ✓

$$\boxed{x = 3}$$

(b) $\dfrac{10}{x} = 5$
Step 1: NPV: $x \neq 0$.
Step 2: Multiply both sides by x.

$$10 = 5x \quad \Rightarrow \quad x = 2$$

Step 3: $x = 2 \neq 0$. Valid.
Step 4: Verify: $\dfrac{10}{2} = 5$ ✓

$$\boxed{x = 2}$$

(c) $\dfrac{4}{x+2} = 1$
Step 1: NPV: $x \neq -2$.
Step 2: Multiply both sides by $(x+2)$.

$$4 = 1(x+2) \quad \Rightarrow \quad 4 = x + 2 \quad \Rightarrow \quad x = 2$$

Step 3: $x = 2 \neq -2$. Valid.
Step 4: Verify: $\dfrac{4}{2+2} = \dfrac{4}{4} = 1$ ✓

$$\boxed{x = 2}$$

(d) $\dfrac{3}{2x-4} = 1$
Step 1: NPV: $2x - 4 = 0 \Rightarrow x = 2$, so $x \neq 2$.
Step 2: Multiply both sides by $(2x-4)$.

$$3 = 2x - 4 \quad \Rightarrow \quad 7 = 2x \quad \Rightarrow \quad x = \frac{7}{2}$$

Step 3: $x = \frac{7}{2} \neq 2$. Valid.
Step 4: Verify: $\frac{3}{2(\frac{7}{2}) - 4} = \frac{3}{7-4} = \frac{3}{3} = 1$ ✓

$$\boxed{x = \frac{7}{2}}$$

2.
(a) $\frac{x}{4} = \frac{3}{2}$
Step 1: Cross-multiply.

$$2x = 12 \quad \Rightarrow \quad x = 6$$

Step 2: Verify: $\frac{6}{4} = \frac{3}{2}$ ✓

$$\boxed{x = 6}$$

(b) $\frac{x+1}{5} = \frac{2x-1}{3}$
Step 1: Cross-multiply.

$$3(x+1) = 5(2x-1)$$

$$3x + 3 = 10x - 5$$

Step 2: Solve.

$$3 + 5 = 10x - 3x \quad \Rightarrow \quad 8 = 7x \quad \Rightarrow \quad x = \frac{8}{7}$$

Step 3: Verify: $\frac{\frac{8}{7}+1}{5} = \frac{\frac{15}{7}}{5} = \frac{15}{35} = \frac{3}{7}$ and $\frac{2(\frac{8}{7})-1}{3} = \frac{\frac{16}{7}-1}{3} = \frac{\frac{9}{7}}{3} = \frac{9}{21} = \frac{3}{7}$ ✓

$$\boxed{x = \frac{8}{7}}$$

(c) $\frac{3}{x-1} = \frac{5}{x+2}$
Step 1: NPVs: $x \neq 1$ and $x \neq -2$.
Step 2: Cross-multiply.

$$3(x+2) = 5(x-1)$$

$$3x + 6 = 5x - 5$$

Step 3: Solve.

$$6 + 5 = 5x - 3x \quad \Rightarrow \quad 11 = 2x \quad \Rightarrow \quad x = \frac{11}{2}$$

Step 4: $x = \frac{11}{2}$ is neither 1 nor -2. Valid.

Step 5: Verify: $\dfrac{3}{\frac{11}{2}-1} = \dfrac{3}{\frac{9}{2}} = \dfrac{6}{9} = \dfrac{2}{3}$ and $\dfrac{5}{\frac{11}{2}+2} = \dfrac{5}{\frac{15}{2}} = \dfrac{10}{15} = \dfrac{2}{3}$ ✓

$$\boxed{x = \frac{11}{2}}$$

3.

(a) $\dfrac{2}{x-3} + \dfrac{1}{x+3} = \dfrac{6}{x^2-9}$

Step 1: Factor. $x^2 - 9 = (x-3)(x+3)$. NPVs: $x \neq 3$, $x \neq -3$.

Step 2: LCD $= (x-3)(x+3)$. Multiply every term.

$$2(x+3) + 1(x-3) = 6$$

Step 3: Expand.

$$2x + 6 + x - 3 = 6$$

$$3x + 3 = 6$$

Step 4: Solve.

$$3x = 3 \quad \Rightarrow \quad x = 1$$

Step 5: $x = 1$ is not an NPV. Valid.

Step 6: Verify: $\dfrac{2}{1-3} + \dfrac{1}{1+3} = \dfrac{2}{-2} + \dfrac{1}{4} = -1 + \dfrac{1}{4} = -\dfrac{3}{4}$ and $\dfrac{6}{1-9} = \dfrac{6}{-8} = -\dfrac{3}{4}$ ✓

$$\boxed{x = 1}$$

(b) $\dfrac{3}{x+2} - \dfrac{1}{x-2} = \dfrac{8}{x^2-4}$

Step 1: Factor. $x^2 - 4 = (x-2)(x+2)$. NPVs: $x \neq 2$, $x \neq -2$.

Step 2: LCD $= (x-2)(x+2)$. Multiply every term.

$$3(x-2) - 1(x+2) = 8$$

Step 3: Expand.

$$3x - 6 - x - 2 = 8$$

$$2x - 8 = 8$$

Step 4: Solve.

$$2x = 16 \quad \Rightarrow \quad x = 8$$

Step 5: $x = 8$ is not an NPV. Valid.

Step 6: Verify: $\dfrac{3}{10} - \dfrac{1}{6} = \dfrac{9}{30} - \dfrac{5}{30} = \dfrac{4}{30} = \dfrac{2}{15}$ and $\dfrac{8}{64-4} = \dfrac{8}{60} = \dfrac{2}{15}$ ✓

$$\boxed{x = 8}$$

4.

(a) $\dfrac{x}{x-4} = \dfrac{4}{x-4} + 2$

Step 1: NPV: $x \neq 4$.

Step 2: LCD $= x - 4$. Multiply every term.

$$x = 4 + 2(x - 4)$$

Step 3: Expand.

$$x = 4 + 2x - 8 = 2x - 4$$

Step 4: Solve.

$$x - 2x = -4 \quad \Rightarrow \quad -x = -4 \quad \Rightarrow \quad x = 4$$

Step 5: $x = 4$ is an NPV. This is an extraneous solution.

The equation has **no solution**.

$$\boxed{\text{No solution}}$$

(b) $\dfrac{x+3}{x} = \dfrac{4}{x} + 1$

Step 1: NPV: $x \neq 0$.

Step 2: LCD $= x$. Multiply every term.

$$x + 3 = 4 + x$$

Step 3: Simplify.

$$3 = 4$$

This is a contradiction. The equation has **no solution**.

$$\boxed{\text{No solution}}$$

5. Solve $\dfrac{x}{x+2} + \dfrac{2}{x-3} = \dfrac{10}{x^2 - x - 6}$.

Step 1: Factor. $x^2 - x - 6 = (x+2)(x-3)$. NPVs: $x \neq -2$, $x \neq 3$.

Step 2: LCD $= (x+2)(x-3)$. Multiply every term.

$$x(x-3) + 2(x+2) = 10$$

Step 3: Expand.

$$x^2 - 3x + 2x + 4 = 10$$

$$x^2 - x + 4 = 10$$

Step 4: Rearrange.

$$x^2 - x - 6 = 0$$

Step 5: Factor.

$$(x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3 \quad \text{or} \quad x = -2$$

Step 6: Both $x = 3$ and $x = -2$ are NPVs. Both solutions are extraneous.
The equation has **no solution**.

$$\boxed{\text{No solution}}$$

6. Solve $\dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{1}{4}$.
Step 1: NPVs: $x \neq 1$, $x \neq -1$.
Step 2: LCD $= 4(x-1)(x+1)$. Multiply every term.

$$4(x+1) - 4(x-1) = (x-1)(x+1)$$

Step 3: Expand left side.

$$4x + 4 - 4x + 4 = 8$$

Step 4: Expand right side.

$$(x-1)(x+1) = x^2 - 1$$

Step 5: Set equal.

$$8 = x^2 - 1 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = 3 \quad \text{or} \quad x = -3$$

Step 6: Neither 3 nor -3 is an NPV. Both are valid candidates.
Step 7: Verify $x = 3$.

$$\frac{1}{3-1} - \frac{1}{3+1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \checkmark$$

Step 8: Verify $x = -3$.

$$\frac{1}{-3-1} - \frac{1}{-3+1} = \frac{1}{-4} - \frac{1}{-2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \checkmark$$

$$\boxed{x = 3 \text{ and } x = -3}$$

7. A student solved $\dfrac{2}{x-1} = \dfrac{3}{x+1}$ and got $x = 5$. Verify and show full solution.
Step 1: NPVs: $x \neq 1$, $x \neq -1$.
Step 2: Cross-multiply.

$$2(x+1) = 3(x-1)$$

$$2x + 2 = 3x - 3$$

Step 3: Solve.

$$2 + 3 = 3x - 2x \quad \Rightarrow \quad x = 5$$

Step 4: $x = 5$ is not an NPV. Valid.
Step 5: Verify: $\dfrac{2}{5-1} = \dfrac{2}{4} = \dfrac{1}{2}$ and $\dfrac{3}{5+1} = \dfrac{3}{6} = \dfrac{1}{2}$ ✓

The student's answer is correct.

$$\boxed{x = 5 \text{ (correct)}}$$

8. Solve $\dfrac{x}{x-2} + \dfrac{x}{x+2} = \dfrac{8}{x^2-4}$.
Step 1: Factor. $x^2 - 4 = (x-2)(x+2)$. NPVs: $x \neq 2$, $x \neq -2$.
Step 2: LCD $= (x-2)(x+2)$. Multiply every term.

$$x(x+2) + x(x-2) = 8$$

Step 3: Expand.

$$x^2 + 2x + x^2 - 2x = 8$$

$$2x^2 = 8$$

Step 4: Solve.

$$x^2 = 4 \quad \Rightarrow \quad x = 2 \quad \text{or} \quad x = -2$$

Step 5: Both $x = 2$ and $x = -2$ are NPVs. Both are extraneous.
The equation has **no solution**.

$$\boxed{\text{No solution}}$$

9. Solve $\dfrac{3}{x+2} + \dfrac{2}{x-3} = \dfrac{5x-3}{x^2-x-6}$.
Step 1: Factor. $x^2 - x - 6 = (x+2)(x-3)$. NPVs: $x \neq -2$, $x \neq 3$.
Step 2: LCD $= (x+2)(x-3)$. Multiply every term.

$$3(x-3) + 2(x+2) = 5x - 3$$

Step 3: Expand the left side.

$$3x - 9 + 2x + 4 = 5x - 3$$

$$5x - 5 = 5x - 3$$

Step 4: Simplify.

$$-5 = -3$$

This is a contradiction. The equation has **no solution**.

$$\boxed{\text{No solution}}$$

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