Unit 2 Review — Rational Expressions and Equations
ApexMath

Lesson Summaries

Lesson 1 — What is a Rational Expression?

A rational expression is a fraction where the numerator, denominator, or both are polynomials. The denominator can never equal zero — values that make the denominator zero are called **non-permissible values** (NPVs).

To find NPVs: factor the denominator, set each factor equal to zero, and solve.

To simplify: factor the numerator and denominator completely, state NPVs from the original denominator, then cancel common factors. You can only cancel **factors**, not terms.

$$\frac{x-a}{a-x} = -1, \quad x \neq a$$

Lesson 2 — Multiplying and Dividing Rational Expressions

Multiply by multiplying numerators together and denominators together. Factor everything first, then cancel common factors before or after multiplying.

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$$

Divide by multiplying by the reciprocal of the second fraction. When flipping the second fraction, its original numerator becomes a denominator — add those NPVs too.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$$

Always collect NPVs from every denominator that appears at any stage.

Lesson 3 — Adding and Subtracting Rational Expressions

You can only add or subtract fractions with the same denominator. If the denominators differ, find the LCD by factoring all denominators and taking each unique factor to its highest power.

Rewrite each fraction with the LCD, then add or subtract the numerators. Distribute the subtraction sign through the entire numerator being subtracted. Simplify the result and check for cancellation.

Lesson 4 — Solving Rational Equations

To solve a rational equation, multiply every term on both sides by the LCD to eliminate all denominators. Solve the resulting polynomial equation. Always:

- State NPVs **before** solving.
- Reject any solution that equals an NPV (extraneous solution).

- Verify valid solutions in the original equation.

For proportions $\frac{A}{B} = \frac{C}{D}$, use cross-multiplication: $AD = BC$.

If eliminating denominators produces a quadratic, solve by factoring or the quadratic formula, then check both solutions.

Review Practice

Section A — Simplifying Rational Expressions

1. State all non-permissible values for each expression.

   (a) $\dfrac{4}{x-7}$

   (b) $\dfrac{x+3}{x^2-5x+6}$

   (c) $\dfrac{2x}{x^2-x-12}$

2. Simplify each expression. State all NPVs.

   (a) $\dfrac{x^2-36}{x+6}$

   (b) $\dfrac{x^2+7x+12}{x^2-x-12}$

   (c) $\dfrac{3x^2-3x}{x^2+2x-3}$

(d) $\dfrac{x-4}{4-x}$

Section B — Multiplying and Dividing

3. Multiply and simplify. State all NPVs.

(a) $\dfrac{4x}{x+3} \times \dfrac{x^2-9}{8x^2}$

(b) $\dfrac{x^2-4}{x+1} \times \dfrac{x^2-1}{x-2}$

(c) $\dfrac{x^2+x-6}{x^2-9} \times \dfrac{x^2-x-6}{x^2-4}$

4. Divide and simplify. State all NPVs.

(a) $\dfrac{x^2-1}{x+2} \div \dfrac{x-1}{x+2}$

(b) $\dfrac{x^2+5x+6}{x^2-4} \div \dfrac{x+3}{x-2}$

(c) $\dfrac{2x^2-18}{x^2-x-6} \div \dfrac{x+3}{x-3}$

Section C — Adding and Subtracting

5. Add or subtract. State all NPVs.

(a) $\dfrac{3x+1}{x-2} - \dfrac{x+3}{x-2}$

(b) $\dfrac{4}{x+3} + \dfrac{2}{x-1}$

(c) $\dfrac{x}{x^2-4} + \dfrac{3}{x+2}$

(d) $\dfrac{2}{x^2-x-6} - \dfrac{1}{x-3}$

Section D — Solving Rational Equations

6. Solve each equation. State NPVs and check for extraneous solutions.

(a) $\dfrac{5}{x} = \dfrac{1}{3}$

(b) $\dfrac{x+2}{4} = \dfrac{3x-1}{6}$

(c) $\dfrac{3}{x-1} + \dfrac{2}{x+1} = \dfrac{10}{x^2-1}$

(d) $\dfrac{x}{x+3} - \dfrac{3}{x-3} = \dfrac{18}{x^2-9}$

Section E — Challenge Problems

7. Simplify $\dfrac{2x^2+5x-3}{2x^2-7x+3}$. State all NPVs.
   Hint: $2x^2+5x-3 = (2x-1)(x+3)$ *and* $2x^2-7x+3 = (2x-1)(x-3)$.

8. Simplify $\dfrac{x^2+3x}{x^2-9} \times \dfrac{x^2-6x+9}{x^2+x-6}$. State all NPVs.

9. Solve $\dfrac{1}{x-2} - \dfrac{1}{x+2} = \dfrac{1}{3}$. State all NPVs and check both solutions.

Answer Key

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Section A — Simplifying Rational Expressions

1.

(a) $\dfrac{4}{x-7}$

Step 1: Set denominator equal to zero. $x - 7 = 0 \Rightarrow x = 7$.

$$\boxed{x \neq 7}$$

(b) $\dfrac{x+3}{x^2-5x+6}$

Step 1: Factor denominator. $x^2 - 5x + 6 = (x-2)(x-3)$.

Step 2: Set each factor to zero.

$$x = 2 \quad \text{and} \quad x = 3$$

$$\boxed{x \neq 2 \text{ and } x \neq 3}$$

(c) $\dfrac{2x}{x^2-x-12}$

Step 1: Factor denominator. $x^2 - x - 12 = (x-4)(x+3)$.

Step 2: Set each factor to zero.

$$x = 4 \quad \text{and} \quad x = -3$$

$$\boxed{x \neq 4 \text{ and } x \neq -3}$$

2.

(a) $\dfrac{x^2-36}{x+6}$

Step 1: NPV: $x \neq -6$.

Step 2: Factor numerator. $x^2 - 36 = (x-6)(x+6)$.

Step 3: Cancel $(x+6)$.

$$\frac{(x-6)(x+6)}{(x+6)} = x - 6$$

$$\boxed{x - 6, \quad x \neq -6}$$

(b) $\dfrac{x^2+7x+12}{x^2-x-12}$

Step 1: Factor numerator. $x^2 + 7x + 12 = (x+3)(x+4)$.

Step 2: Factor denominator. $x^2 - x - 12 = (x-4)(x+3)$.

Step 3: NPVs: $x \neq 4$ and $x \neq -3$.

Step 4: Cancel $(x+3)$.

$$\frac{(x+3)(x+4)}{(x-4)(x+3)} = \frac{x+4}{x-4}$$

$$\boxed{\frac{x+4}{x-4}, \quad x \neq 4,\ x \neq -3}$$

(c) $\dfrac{3x^2-3x}{x^2+2x-3}$

Step 1: Factor numerator. $3x^2-3x = 3x(x-1)$.
Step 2: Factor denominator. $x^2+2x-3 = (x+3)(x-1)$.
Step 3: NPVs: $x \neq -3$ and $x \neq 1$.
Step 4: Cancel $(x-1)$.

$$\frac{3x(x-1)}{(x+3)(x-1)} = \frac{3x}{x+3}$$

$$\boxed{\frac{3x}{x+3}, \quad x \neq 1,\ x \neq -3}$$

(d) $\dfrac{x-4}{4-x}$

Step 1: NPV: $4-x=0 \Rightarrow x=4$, so $x \neq 4$.
Step 2: Rewrite numerator. $x-4 = -(4-x)$.

$$\frac{-(4-x)}{4-x} = -1$$

$$\boxed{-1, \quad x \neq 4}$$

Section B — Multiplying and Dividing

3.

(a) $\dfrac{4x}{x+3} \times \dfrac{x^2-9}{8x^2}$

Step 1: Factor. $x^2-9 = (x-3)(x+3)$.
Step 2: NPVs: $x \neq -3$ and $x \neq 0$.
Step 3: Cancel $(x+3)$, one x, and reduce $4/8$.

$$\frac{4x}{(x+3)} \times \frac{(x-3)(x+3)}{8x \cdot x} = \frac{4(x-3)}{8x} = \frac{x-3}{2x}$$

$$\boxed{\frac{x-3}{2x}, \quad x \neq -3,\ x \neq 0}$$

(b) $\dfrac{x^2-4}{x+1} \times \dfrac{x^2-1}{x-2}$

Step 1: Factor. $x^2-4 = (x-2)(x+2)$, $x^2-1 = (x-1)(x+1)$.

Step 2: NPVs: $x \neq -1$ and $x \neq 2$.
Step 3: Cancel $(x+1)$ and $(x-2)$.

$$\frac{(x-2)(x+2)}{(x+1)} \times \frac{(x-1)(x+1)}{(x-2)} = (x+2)(x-1)$$

$$\boxed{(x+2)(x-1), \quad x \neq -1,\ x \neq 2}$$

(c) $\dfrac{x^2+x-6}{x^2-9} \times \dfrac{x^2-x-6}{x^2-4}$
Step 1: Factor everything.

$$x^2+x-6 = (x+3)(x-2) \qquad x^2-9 = (x-3)(x+3)$$

$$x^2-x-6 = (x-3)(x+2) \qquad x^2-4 = (x-2)(x+2)$$

Step 2: NPVs: $x \neq 3$, $x \neq -3$, $x \neq 2$, $x \neq -2$.
Step 3: Write all factors and cancel $(x+3)$, $(x-2)$, $(x-3)$, $(x+2)$.

$$\frac{(x+3)(x-2)}{(x-3)(x+3)} \times \frac{(x-3)(x+2)}{(x-2)(x+2)} = 1$$

$$\boxed{1, \quad x \neq 3,\ x \neq -3,\ x \neq 2,\ x \neq -2}$$

4.

(a) $\dfrac{x^2-1}{x+2} \div \dfrac{x-1}{x+2}$
Step 1: Flip and multiply.

$$\frac{x^2-1}{x+2} \times \frac{x+2}{x-1}$$

Step 2: Factor. $x^2-1 = (x-1)(x+1)$.
Step 3: NPVs: $x \neq -2$ and $x \neq 1$.
Step 4: Cancel $(x+2)$ and $(x-1)$.

$$\frac{(x-1)(x+1)}{(x+2)} \times \frac{(x+2)}{(x-1)} = \frac{(x-1)(x+1)}{(x-1)} = x+1$$

$$\boxed{x+1, \quad x \neq -2,\ x \neq 1}$$

(b) $\dfrac{x^2+5x+6}{x^2-4} \div \dfrac{x+3}{x-2}$
Step 1: Flip and multiply.

$$\frac{x^2+5x+6}{x^2-4} \times \frac{x-2}{x+3}$$

Step 2: Factor. $x^2+5x+6 = (x+2)(x+3)$, $x^2-4 = (x-2)(x+2)$.
Step 3: NPVs: $x \neq 2$, $x \neq -2$, $x \neq -3$.

Step 4: Cancel $(x+3)$, $(x+2)$, and $(x-2)$.

$$\frac{(x+2)(x+3)}{(x-2)(x+2)} \times \frac{(x-2)}{(x+3)} = 1$$

$$\boxed{1, \quad x \neq 2,\ x \neq -2,\ x \neq -3}$$

(c) $\dfrac{2x^2-18}{x^2-x-6} \div \dfrac{x+3}{x-3}$

Step 1: Flip and multiply.

$$\frac{2x^2-18}{x^2-x-6} \times \frac{x-3}{x+3}$$

Step 2: Factor. $2x^2-18 = 2(x^2-9) = 2(x-3)(x+3)$, $x^2-x-6 = (x-3)(x+2)$.

Step 3: NPVs: $x \neq 3$, $x \neq -2$, $x \neq -3$.

Step 4: Cancel $(x-3)$ and $(x+3)$.

$$\frac{2(x-3)(x+3)}{(x-3)(x+2)} \times \frac{(x-3)}{(x+3)}$$

Wait — after the flip, we have $(x-3)$ appearing in both numerators. Let us count carefully.

Numerator factors: 2, $(x-3)$, $(x+3)$, $(x-3)$

Denominator factors: $(x-3)$, $(x+2)$, $(x+3)$

Cancel one $(x-3)$ from numerator and denominator, and cancel $(x+3)$:

$$\frac{2(x-3)}{x+2}$$

$$\boxed{\frac{2(x-3)}{x+2}, \quad x \neq 3,\ x \neq -2,\ x \neq -3}$$

Section C — Adding and Subtracting

5.

(a) $\dfrac{3x+1}{x-2} - \dfrac{x+3}{x-2}$

Step 1: Same denominator. NPV: $x \neq 2$.

Step 2: Subtract numerators with brackets.

$$\frac{(3x+1)-(x+3)}{x-2} = \frac{3x+1-x-3}{x-2} = \frac{2x-2}{x-2}$$

Step 3: Factor numerator. $2x-2 = 2(x-1)$. No cancellation.

$$\boxed{\frac{2(x-1)}{x-2}, \quad x \neq 2}$$

(b) $\dfrac{4}{x+3} + \dfrac{2}{x-1}$

Step 1: NPVs: $x \neq -3$ and $x \neq 1$.

Step 2: LCD $= (x+3)(x-1)$.

Step 3: Rewrite and add.

$$\frac{4(x-1)}{(x+3)(x-1)} + \frac{2(x+3)}{(x+3)(x-1)} = \frac{4x-4+2x+6}{(x+3)(x-1)} = \frac{6x+2}{(x+3)(x-1)}$$

Step 4: Factor numerator. $6x+2 = 2(3x+1)$.

$$\boxed{\frac{2(3x+1)}{(x+3)(x-1)}, \quad x \neq -3,\ x \neq 1}$$

(c) $\dfrac{x}{x^2-4} + \dfrac{3}{x+2}$

Step 1: Factor. $x^2-4 = (x-2)(x+2)$. NPVs: $x \neq 2$, $x \neq -2$.

Step 2: LCD $= (x-2)(x+2)$.

Step 3: Rewrite second fraction.

$$\frac{x}{(x-2)(x+2)} + \frac{3(x-2)}{(x-2)(x+2)} = \frac{x+3(x-2)}{(x-2)(x+2)} = \frac{x+3x-6}{(x-2)(x+2)} = \frac{4x-6}{(x-2)(x+2)}$$

Step 4: Factor numerator. $4x-6 = 2(2x-3)$. No cancellation.

$$\boxed{\frac{2(2x-3)}{(x-2)(x+2)}, \quad x \neq 2,\ x \neq -2}$$

(d) $\dfrac{2}{x^2-x-6} - \dfrac{1}{x-3}$

Step 1: Factor. $x^2-x-6 = (x-3)(x+2)$. NPVs: $x \neq 3$, $x \neq -2$.

Step 2: LCD $= (x-3)(x+2)$.

Step 3: Rewrite second fraction.

$$\frac{2}{(x-3)(x+2)} - \frac{1(x+2)}{(x-3)(x+2)} = \frac{2-(x+2)}{(x-3)(x+2)} = \frac{2-x-2}{(x-3)(x+2)} = \frac{-x}{(x-3)(x+2)}$$

$$\boxed{\frac{-x}{(x-3)(x+2)}, \quad x \neq 3,\ x \neq -2}$$

Section D — Solving Rational Equations

6.

(a) $\dfrac{5}{x} = \dfrac{1}{3}$

Step 1: NPV: $x \neq 0$.

Step 2: Cross-multiply.

$$5 \times 3 = 1 \times x \quad \Rightarrow \quad x = 15$$

Step 3: $x = 15 \neq 0$. Valid. Verify: $\dfrac{5}{15} = \dfrac{1}{3}$ ✓

$$\boxed{x = 15}$$

(b) $\dfrac{x+2}{4} = \dfrac{3x-1}{6}$

Step 1: No variable in denominators, no NPVs.

Step 2: Cross-multiply.

$$6(x+2) = 4(3x-1)$$

$$6x + 12 = 12x - 4$$

Step 3: Solve.

$$12 + 4 = 12x - 6x \quad \Rightarrow \quad 16 = 6x \quad \Rightarrow \quad x = \frac{8}{3}$$

Step 4: Verify: $\dfrac{\frac{8}{3}+2}{4} = \dfrac{\frac{14}{3}}{4} = \dfrac{14}{12} = \dfrac{7}{6}$ and $\dfrac{3(\frac{8}{3})-1}{6} = \dfrac{8-1}{6} = \dfrac{7}{6}$ ✓

$$\boxed{x = \frac{8}{3}}$$

(c) $\dfrac{3}{x-1} + \dfrac{2}{x+1} = \dfrac{10}{x^2-1}$

Step 1: Factor. $x^2 - 1 = (x-1)(x+1)$. NPVs: $x \neq 1$, $x \neq -1$.

Step 2: LCD $= (x-1)(x+1)$. Multiply every term.

$$3(x+1) + 2(x-1) = 10$$

$$3x + 3 + 2x - 2 = 10$$

$$5x + 1 = 10 \quad \Rightarrow \quad 5x = 9 \quad \Rightarrow \quad x = \frac{9}{5}$$

Step 3: $x = \dfrac{9}{5}$ is not an NPV. Valid.

Step 4: Verify: $\dfrac{3}{\frac{9}{5}-1} + \dfrac{2}{\frac{9}{5}+1} = \dfrac{3}{\frac{4}{5}} + \dfrac{2}{\frac{14}{5}} = \dfrac{15}{4} + \dfrac{10}{14} = \dfrac{15}{4} + \dfrac{5}{7} = \dfrac{105}{28} + \dfrac{20}{28} = \dfrac{125}{28}$

Check right side: $\dfrac{10}{(\frac{9}{5})^2 - 1} = \dfrac{10}{\frac{81}{25} - 1} = \dfrac{10}{\frac{56}{25}} = \dfrac{250}{56} = \dfrac{125}{28}$ ✓

$$\boxed{x = \frac{9}{5}}$$

(d) $\dfrac{x}{x+3} - \dfrac{3}{x-3} = \dfrac{18}{x^2-9}$

Step 1: Factor. $x^2 - 9 = (x-3)(x+3)$. NPVs: $x \neq 3$, $x \neq -3$.
Step 2: LCD $= (x-3)(x+3)$. Multiply every term.

$$x(x-3) - 3(x+3) = 18$$

Step 3: Expand.

$$x^2 - 3x - 3x - 9 = 18$$

$$x^2 - 6x - 9 = 18$$

Step 4: Rearrange.

$$x^2 - 6x - 27 = 0$$

Step 5: Factor.

$$(x-9)(x+3) = 0 \quad \Rightarrow \quad x = 9 \quad \text{or} \quad x = -3$$

Step 6: $x = -3$ is an NPV. Reject it. $x = 9$ is valid.
Step 7: Verify $x = 9$: $\dfrac{9}{12} - \dfrac{3}{6} = \dfrac{3}{4} - \dfrac{1}{2} = \dfrac{1}{4}$ and $\dfrac{18}{81-9} = \dfrac{18}{72} = \dfrac{1}{4}$ ✓

$$\boxed{x = 9}$$

Section E — Challenge Problems

7. Simplify $\dfrac{2x^2 + 5x - 3}{2x^2 - 7x + 3}$.
Step 1: Use the hint. $2x^2 + 5x - 3 = (2x-1)(x+3)$ and $2x^2 - 7x + 3 = (2x-1)(x-3)$.
Step 2: NPVs: $(2x-1) = 0 \Rightarrow x = \dfrac{1}{2}$ and $(x-3) = 0 \Rightarrow x = 3$.
Step 3: Cancel $(2x-1)$.

$$\frac{(2x-1)(x+3)}{(2x-1)(x-3)} = \frac{x+3}{x-3}$$

$$\boxed{\frac{x+3}{x-3}, \quad x \neq \frac{1}{2},\ x \neq 3}$$

8. Simplify $\dfrac{x^2 + 3x}{x^2 - 9} \times \dfrac{x^2 - 6x + 9}{x^2 + x - 6}$.
Step 1: Factor everything.

$$x^2 + 3x = x(x+3)$$

$$x^2 - 9 = (x-3)(x+3)$$

$$x^2 - 6x + 9 = (x-3)^2$$

$$x^2 + x - 6 = (x+3)(x-2)$$

Step 2: NPVs from all denominators: $x \neq 3$, $x \neq -3$, $x \neq 2$.
Step 3: Write all factors and cancel.
Numerators: x, $(x+3)$, $(x-3)^2$

Denominators: $(x-3)$, $(x+3)$, $(x+3)$, $(x-2)$
Cancel one $(x+3)$ and one $(x-3)$:

$$\frac{x\cdot(x+3)\cdot(x-3)^2}{(x-3)\cdot(x+3)\cdot(x+3)\cdot(x-2)} = \frac{x(x-3)}{(x+3)(x-2)}$$

$$\boxed{\frac{x(x-3)}{(x+3)(x-2)}, \quad x\neq 3,\ x\neq -3,\ x\neq 2}$$

9. Solve $\dfrac{1}{x-2} - \dfrac{1}{x+2} = \dfrac{1}{3}$.
Step 1: NPVs: $x\neq 2$ and $x\neq -2$.
Step 2: LCD $= 3(x-2)(x+2)$. Multiply every term.

$$3(x+2) - 3(x-2) = (x-2)(x+2)$$

Step 3: Expand left side.

$$3x+6-3x+6 = 12$$

Step 4: Expand right side.

$$(x-2)(x+2) = x^2-4$$

Step 5: Set equal and solve.

$$x^2-4=12 \quad\Rightarrow\quad x^2=16 \quad\Rightarrow\quad x=4 \quad\text{or}\quad x=-4$$

Step 6: Neither 4 nor -4 is an NPV. Both are valid candidates.
Step 7: Verify $x=4$.

$$\frac{1}{4-2} - \frac{1}{4+2} = \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}\checkmark$$

Step 8: Verify $x=-4$.

$$\frac{1}{-4-2} - \frac{1}{-4+2} = \frac{1}{-6} - \frac{1}{-2} = -\frac{1}{6} + \frac{1}{2} = -\frac{1}{6} + \frac{3}{6} = \frac{2}{6} = \frac{1}{3}\checkmark$$

$$\boxed{x=4 \text{ and } x=-4}$$

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