What is a Rational Expression?
ApexMath

Notes

The Big Idea

A **rational expression** is a fraction where the numerator, the denominator, or both are polynomials. A polynomial is any expression made up of variables and whole number exponents, such as $x+3$, x^2-4, or $2x^2+5x-3$.

Here are some examples of rational expressions:

$$\frac{x+2}{x-5} \qquad \frac{3x}{x^2-9} \qquad \frac{x^2-1}{x+1} \qquad \frac{5}{x}$$

The word *rational* comes from the word *ratio* — a rational expression is simply a ratio of two polynomials.

Non-Permissible Values

The most important rule for rational expressions is:

The denominator can never equal zero.

Division by zero is undefined in mathematics. So any value of x that makes the denominator equal zero is **not allowed**. These are called **non-permissible values** (NPVs), sometimes also called restrictions.

To find the non-permissible values:

- Step 1: Set the denominator equal to zero.
- Step 2: Solve for x.
- Step 3: State that x cannot equal those values.

For example, for $\dfrac{x+2}{x-5}$:

$$x-5=0 \quad \Rightarrow \quad x=5$$

So $x \neq 5$.

When the Denominator has Two Factors

When the denominator is a product of two or more factors, set each factor equal to zero separately.

For example, for $\dfrac{3x}{(x-2)(x+4)}$:

$$x-2=0 \quad \Rightarrow \quad x=2$$

$$x+4=0 \quad \Rightarrow \quad x=-4$$

So $x \neq 2$ and $x \neq -4$.

When the Denominator Must Be Factored First

Sometimes the denominator is not already factored. You must factor it first before finding NPVs.

For example, for $\dfrac{x+1}{x^2-x-6}$:

Step 1: Factor the denominator.

$$x^2 - x - 6 = (x-3)(x+2)$$

Step 2: Set each factor equal to zero.

$$x - 3 = 0 \quad \Rightarrow \quad x = 3$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2$$

So $x \neq 3$ and $x \neq -2$.

Simplifying Rational Expressions

Simplifying a rational expression means cancelling common factors that appear in both the numerator and the denominator. This is exactly the same as simplifying a numerical fraction like $\frac{6}{9} = \frac{2}{3}$ by dividing by 3.

The key rule is:

You can only cancel factors, not terms.

A **factor** is something being multiplied. A **term** is something being added or subtracted. For example:

$$\frac{x(x+3)}{x} = x + 3 \quad \checkmark \qquad (x \text{ is a factor in both})$$

$$\frac{x+3}{x} \neq 3 \quad \times \qquad (x \text{ is a term, not a factor, in the numerator})$$

The Method for Simplifying

- Step 1: Completely factor both the numerator and the denominator.
- Step 2: State the non-permissible values using the *original* denominator before cancelling.
- Step 3: Cancel any factors that appear in both the numerator and denominator.
- Step 4: Write the simplified expression.

It is critical to find the NPVs from the original denominator. After cancelling, a restriction might disappear from the expression, but it still applies.

Example of Simplifying

Simplify $\dfrac{x^2-9}{x+3}$.

Step 1: Factor the numerator. $x^2 - 9 = (x-3)(x+3)$

Step 2: State the NPV from the original denominator: $x + 3 = 0 \Rightarrow x \neq -3$

Step 3: Cancel the common factor $(x + 3)$.

$$\frac{(x-3)(x+3)}{(x+3)} = x - 3$$

The simplified expression is $x - 3$, where $x \neq -3$.

Pro tip

- Always find the non-permissible values **before** you simplify. Once you cancel a factor, it is gone — you might forget the restriction it carried.
- You can only cancel **factors**. If the numerator or denominator has addition or subtraction, you must factor first before anything can be cancelled.
- A common mistake is cancelling across a plus or minus sign. $\frac{x+6}{x} \neq 6$. The x in the numerator is a term, not a factor.
- When you state your final answer, always include the non-permissible value(s) alongside the simplified expression.

Examples

Example A

State the non-permissible values of $\frac{5}{x+7}$.

Step 1: Set the denominator equal to zero.

$$x + 7 = 0$$

Step 2: Solve.

$$x = -7$$

The non-permissible value is $x \neq -7$.

Example B

State the non-permissible values of $\frac{2x}{x^2 - 5x + 6}$.

Step 1: Factor the denominator.

$$x^2 - 5x + 6 = (x-2)(x-3)$$

Step 2: Set each factor equal to zero.

$$x - 2 = 0 \quad \Rightarrow \quad x = 2$$

$$x - 3 = 0 \quad \Rightarrow \quad x = 3$$

The non-permissible values are $x \neq 2$ and $x \neq 3$.

Example C

Simplify $\dfrac{3x+12}{x^2+4x}$. State all non-permissible values.

Step 1: Factor the numerator.

$$3x + 12 = 3(x + 4)$$

Step 2: Factor the denominator.

$$x^2 + 4x = x(x + 4)$$

Step 3: Find the NPVs from the original denominator $x(x + 4)$.

$$x = 0 \qquad \text{and} \qquad x + 4 = 0 \Rightarrow x = -4$$

So $x \neq 0$ and $x \neq -4$.

Step 4: Cancel the common factor $(x + 4)$.

$$\frac{3(x+4)}{x(x+4)} = \frac{3}{x}$$

The simplified expression is $\dfrac{3}{x}$, where $x \neq 0$ and $x \neq -4$.

Example D

Simplify $\dfrac{x^2-16}{x^2-x-12}$. State all non-permissible values.

Step 1: Factor the numerator. $x^2 - 16$ is a difference of squares.

$$x^2 - 16 = (x - 4)(x + 4)$$

Step 2: Factor the denominator.

$$x^2 - x - 12 = (x - 4)(x + 3)$$

Step 3: Find the NPVs from the original denominator.

$$x - 4 = 0 \Rightarrow x = 4 \qquad \text{and} \qquad x + 3 = 0 \Rightarrow x = -3$$

So $x \neq 4$ and $x \neq -3$.

Step 4: Cancel the common factor $(x - 4)$.

$$\frac{(x-4)(x+4)}{(x-4)(x+3)} = \frac{x+4}{x+3}$$

The simplified expression is $\dfrac{x+4}{x+3}$, where $x \neq 4$ and $x \neq -3$.

Common mistakes

- **Cancelling terms instead of factors.** You cannot cancel x in $\frac{x+5}{x}$. The x in the numerator is part of a sum, not a separate factor.
- **Forgetting to state NPVs.** Non-permissible values are part of the answer. A simplified expression without its restrictions is an incomplete answer.
- **Finding NPVs after simplifying.** Always identify restrictions from the original denominator before cancelling anything.
- **Factoring errors.** A mistake in factoring leads to wrong cancellations. Always double-check your factoring by expanding back out.
- **Thinking $\frac{x-3}{3-x} = 1$.** Notice that $x - 3 = -(3 - x)$. So $\frac{x-3}{3-x} = \frac{-(3-x)}{3-x} = -1$, not 1.

Worksheet

1. State the non-permissible value(s) for each rational expression.

   (a) $\frac{3}{x-4}$

   (b) $\frac{x+1}{x+9}$

   (c) $\frac{2x}{x}$

   (d) $\frac{7}{3x-12}$

2. Factor the denominator, then state all non-permissible values.

   (a) $\frac{x+3}{x^2-4}$

(b) $\dfrac{5}{x^2 + 3x}$

(c) $\dfrac{x - 1}{x^2 - 7x + 12}$

(d) $\dfrac{2x + 1}{x^2 - x - 20}$

3. Simplify each rational expression. State all non-permissible values.

   (a) $\dfrac{5x}{10x^2}$

   (b) $\dfrac{4(x + 2)}{8(x + 2)}$

   (c) $\dfrac{x^2 + 5x}{x}$

   (d) $\dfrac{6x - 18}{3}$

4. Simplify each rational expression. State all non-permissible values.

(a) $\dfrac{x^2 - 25}{x + 5}$

(b) $\dfrac{x^2 + 6x + 9}{x + 3}$

(c) $\dfrac{x^2 - 4}{x^2 - x - 2}$

(d) $\dfrac{x^2 + 2x - 8}{x^2 - x - 6}$

5. Simplify $\dfrac{2x^2 + 6x}{x^2 + 5x + 6}$. State all non-permissible values.

6. Simplify $\dfrac{x^2 - 5x + 6}{x^2 - 4x + 4}$. State all non-permissible values.

7. Simplify $\dfrac{x - 5}{5 - x}$. State all non-permissible values.

8. A student simplified $\dfrac{x^2+3x}{x^2-9}$ and got $\dfrac{x}{x-3}$ with $x \neq 3$. Find and correct any errors.

9. **Challenge:** Simplify $\dfrac{2x^2-8}{x^2+x-6}$. State all non-permissible values.

10. **Challenge:** Simplify $\dfrac{3x^2-12x+12}{x^2-4x+4}$. State all non-permissible values.

Answer Key

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1.

(a) $\dfrac{3}{x-4}$

Step 1: Set denominator equal to zero.

$$x - 4 = 0 \quad \Rightarrow \quad x = 4$$

$$\boxed{x \neq 4}$$

(b) $\dfrac{x+1}{x+9}$

Step 1: Set denominator equal to zero.

$$x + 9 = 0 \quad \Rightarrow \quad x = -9$$

$$\boxed{x \neq -9}$$

(c) $\dfrac{2x}{x}$

Step 1: Set denominator equal to zero.

$$x = 0$$

$$\boxed{x \neq 0}$$

(d) $\dfrac{7}{3x-12}$

Step 1: Set denominator equal to zero.

$$3x - 12 = 0 \quad \Rightarrow \quad 3x = 12 \quad \Rightarrow \quad x = 4$$

$$\boxed{x \neq 4}$$

2.

(a) $\dfrac{x+3}{x^2-4}$

Step 1: Factor the denominator.

$$x^2 - 4 = (x-2)(x+2)$$

Step 2: Set each factor equal to zero.

$$x = 2 \qquad \text{and} \qquad x = -2$$

$$\boxed{x \neq 2 \text{ and } x \neq -2}$$

(b) $\dfrac{5}{x^2+3x}$
Step 1: Factor the denominator.

$$x^2 + 3x = x(x+3)$$

Step 2: Set each factor equal to zero.

$$x = 0 \qquad \text{and} \qquad x + 3 = 0 \Rightarrow x = -3$$

$$\boxed{x \neq 0 \text{ and } x \neq -3}$$

(c) $\dfrac{x-1}{x^2-7x+12}$
Step 1: Factor the denominator.

$$x^2 - 7x + 12 = (x-3)(x-4)$$

Step 2: Set each factor equal to zero.

$$x = 3 \qquad \text{and} \qquad x = 4$$

$$\boxed{x \neq 3 \text{ and } x \neq 4}$$

(d) $\dfrac{2x+1}{x^2-x-20}$
Step 1: Factor the denominator.

$$x^2 - x - 20 = (x-5)(x+4)$$

Step 2: Set each factor equal to zero.

$$x = 5 \qquad \text{and} \qquad x = -4$$

$$\boxed{x \neq 5 \text{ and } x \neq -4}$$

3.

(a) $\dfrac{5x}{10x^2}$
Step 1: NPV from denominator: $10x^2 = 0 \Rightarrow x = 0$, so $x \neq 0$.
Step 2: Factor and cancel.

$$\frac{5x}{10x^2} = \frac{5 \cdot x}{5 \cdot 2 \cdot x \cdot x} = \frac{1}{2x}$$

$$\boxed{\frac{1}{2x}, \quad x \neq 0}$$

(b) $\dfrac{4(x+2)}{8(x+2)}$
Step 1: NPV: $8(x+2)=0 \Rightarrow x=-2$, so $x \neq -2$.
Step 2: Cancel the common factor $(x+2)$.

$$\frac{4(x+2)}{8(x+2)} = \frac{4}{8} = \frac{1}{2}$$

$$\boxed{\frac{1}{2}, \quad x \neq -2}$$

(c) $\dfrac{x^2+5x}{x}$
Step 1: NPV: $x=0$, so $x \neq 0$.
Step 2: Factor the numerator.

$$x^2+5x = x(x+5)$$

Step 3: Cancel x.

$$\frac{x(x+5)}{x} = x+5$$

$$\boxed{x+5, \quad x \neq 0}$$

(d) $\dfrac{6x-18}{3}$
Step 1: The denominator is 3, a constant. It is never zero, so there are no NPVs.
Step 2: Factor the numerator.

$$6x-18 = 3(2x-6) = 6(x-3)$$

Step 3: Divide.

$$\frac{6(x-3)}{3} = 2(x-3) = 2x-6$$

$$\boxed{2x-6, \quad \text{no restrictions}}$$

4.

(a) $\dfrac{x^2-25}{x+5}$
Step 1: NPV: $x+5=0 \Rightarrow x=-5$, so $x \neq -5$.
Step 2: Factor the numerator.

$$x^2-25 = (x-5)(x+5)$$

Step 3: Cancel $(x+5)$.

$$\frac{(x-5)(x+5)}{(x+5)} = x-5$$

$$\boxed{x - 5, \quad x \neq -5}$$

(b) $\dfrac{x^2 + 6x + 9}{x + 3}$
Step 1: NPV: $x + 3 = 0 \Rightarrow x = -3$, so $x \neq -3$.
Step 2: Factor the numerator.

$$x^2 + 6x + 9 = (x + 3)^2 = (x + 3)(x + 3)$$

Step 3: Cancel one $(x + 3)$.

$$\frac{(x + 3)(x + 3)}{(x + 3)} = x + 3$$

$$\boxed{x + 3, \quad x \neq -3}$$

(c) $\dfrac{x^2 - 4}{x^2 - x - 2}$
Step 1: Factor the numerator.

$$x^2 - 4 = (x - 2)(x + 2)$$

Step 2: Factor the denominator.

$$x^2 - x - 2 = (x - 2)(x + 1)$$

Step 3: NPVs: $x = 2$ and $x = -1$, so $x \neq 2$ and $x \neq -1$.
Step 4: Cancel $(x - 2)$.

$$\frac{(x - 2)(x + 2)}{(x - 2)(x + 1)} = \frac{x + 2}{x + 1}$$

$$\boxed{\frac{x + 2}{x + 1}, \quad x \neq 2 \text{ and } x \neq -1}$$

(d) $\dfrac{x^2 + 2x - 8}{x^2 - x - 6}$
Step 1: Factor the numerator.

$$x^2 + 2x - 8 = (x + 4)(x - 2)$$

Step 2: Factor the denominator.

$$x^2 - x - 6 = (x - 3)(x + 2)$$

Step 3: NPVs: $x = 3$ and $x = -2$, so $x \neq 3$ and $x \neq -2$.
Step 4: There are no common factors to cancel.

$$\boxed{\frac{(x + 4)(x - 2)}{(x - 3)(x + 2)}, \quad x \neq 3 \text{ and } x \neq -2}$$

5. Simplify $\dfrac{2x^2+6x}{x^2+5x+6}$.
Step 1: Factor the numerator.

$$2x^2+6x = 2x(x+3)$$

Step 2: Factor the denominator.

$$x^2+5x+6 = (x+2)(x+3)$$

Step 3: NPVs: $x=-2$ and $x=-3$, so $x \neq -2$ and $x \neq -3$.
Step 4: Cancel $(x+3)$.

$$\frac{2x(x+3)}{(x+2)(x+3)} = \frac{2x}{x+2}$$

$$\boxed{\frac{2x}{x+2}, \quad x \neq -2 \text{ and } x \neq -3}$$

6. Simplify $\dfrac{x^2-5x+6}{x^2-4x+4}$.
Step 1: Factor the numerator.

$$x^2-5x+6 = (x-2)(x-3)$$

Step 2: Factor the denominator.

$$x^2-4x+4 = (x-2)^2 = (x-2)(x-2)$$

Step 3: NPV: $x=2$, so $x \neq 2$.
Step 4: Cancel one $(x-2)$.

$$\frac{(x-2)(x-3)}{(x-2)(x-2)} = \frac{x-3}{x-2}$$

$$\boxed{\frac{x-3}{x-2}, \quad x \neq 2}$$

7. Simplify $\dfrac{x-5}{5-x}$.
Step 1: NPV: $5-x=0 \Rightarrow x=5$, so $x \neq 5$.
Step 2: Notice that $x-5 = -(5-x)$. Rewrite the numerator.

$$\frac{x-5}{5-x} = \frac{-(5-x)}{5-x}$$

Step 3: Cancel $(5-x)$.

$$\frac{-(5-x)}{5-x} = -1$$

$$\boxed{-1, \quad x \neq 5}$$

8. A student simplified $\dfrac{x^2+3x}{x^2-9}$ and got $\dfrac{x}{x-3}$ with $x \neq 3$.
Step 1: Factor the numerator correctly.

$$x^2 + 3x = x(x+3)$$

Step 2: Factor the denominator correctly.

$$x^2 - 9 = (x-3)(x+3)$$

Step 3: State the NPVs from the original denominator.

$$x = 3 \quad \text{and} \quad x = -3$$

So $x \neq 3$ **and** $x \neq -3$.
Step 4: Cancel $(x+3)$.

$$\frac{x(x+3)}{(x-3)(x+3)} = \frac{x}{x-3}$$

Step 5: Identify the student's errors. The simplified expression $\dfrac{x}{x-3}$ is correct, but the student missed the restriction $x \neq -3$. Both restrictions must be stated.

$$\boxed{\frac{x}{x-3}, \quad x \neq 3 \text{ and } x \neq -3}$$

9. Simplify $\dfrac{2x^2-8}{x^2+x-6}$.
Step 1: Factor the numerator.

$$2x^2 - 8 = 2(x^2-4) = 2(x-2)(x+2)$$

Step 2: Factor the denominator.

$$x^2 + x - 6 = (x+3)(x-2)$$

Step 3: NPVs: $x = -3$ and $x = 2$, so $x \neq -3$ and $x \neq 2$.
Step 4: Cancel $(x-2)$.

$$\frac{2(x-2)(x+2)}{(x+3)(x-2)} = \frac{2(x+2)}{x+3}$$

$$\boxed{\frac{2(x+2)}{x+3}, \quad x \neq 2 \text{ and } x \neq -3}$$

10. Simplify $\dfrac{3x^2-12x+12}{x^2-4x+4}$.

Step 1: Factor the numerator.

$$3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2$$

Step 2: Factor the denominator.

$$x^2 - 4x + 4 = (x-2)^2$$

Step 3: NPV: $x = 2$, so $x \neq 2$.
Step 4: Cancel $(x-2)^2$.

$$\frac{3(x-2)^2}{(x-2)^2} = 3$$

$$\boxed{3, \quad x \neq 2}$$

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