Linear-Quadratic Systems
ApexMath

Notes

What Is a Linear-Quadratic System?

A **linear-quadratic system** is a system where one equation is a line and the other is a parabola. We are looking for every point (x, y) where the line and the parabola **intersect** — where both equations are true at the same time.

Three Possible Outcomes

Unlike two lines, a line and a parabola can intersect in three different ways.

- **Two solutions:** The line cuts through the parabola at two separate points. This is the most common case.
- **One solution:** The line just grazes the parabola at exactly one point. This line is called a **tangent line** to the parabola.
- **No solution:** The line and the parabola never meet. The line passes entirely above or below the parabola.

How to Tell Which Case You Have: The Discriminant

After substituting and rearranging into the form $ax^2 + bx + c = 0$, compute:

$$\Delta = b^2 - 4ac$$

- $\Delta > 0$: two real solutions (two intersection points)
- $\Delta = 0$: one real solution (tangent line)
- $\Delta < 0$: no real solutions (line and parabola do not intersect)

The Method: Substitution

Because one equation is already solved for y (the parabola, written as $y = \ldots$), we use substitution every time.

Step 1: Write both equations so that one is in the form $y =$ quadratic and the other is $y =$ linear.

Step 2: Set the two right-hand sides equal to each other. This eliminates y entirely.

Step 3: Rearrange into the standard form $ax^2 + bx + c = 0$. Move every term to one side.

Step 4: Solve the quadratic equation by factoring or the quadratic formula. Each x-value you find is an x-coordinate of an intersection point.

Step 5: For each x-value, substitute back into the *linear* equation to find the corresponding y-value. Write each answer as an ordered pair (x, y).

Always Check Your Answer

Substitute each solution into *both* original equations to confirm. Because quadratic equations can produce extraneous roots, checking is especially important here.

Pro Tip

- Always substitute back into the **linear** equation to find y. The linear equation is simpler and less likely to cause arithmetic errors than substituting into the quadratic.
- Before solving, move everything to one side and write the equation in descending order: x^2 term first, then x term, then the constant. This keeps the factoring organised.
- Compute the discriminant $\Delta = b^2 - 4ac$ before attempting to solve. If $\Delta < 0$, stop immediately and state there is no solution. There is no need to factor or apply the quadratic formula.
- When factoring, always double-check by expanding your factors back out. A sign error in factoring is the most common mistake in this lesson.
- If a problem gives you two x-values, you must find *two* ordered pairs — one for each x. A single ordered pair is not a complete answer when there are two intersection points.

Examples

Example 1 (Two Solutions): Solve the system.

$$\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$$

Step 1: Set the right-hand sides equal.

Both equations are solved for y, so set them equal:

$$x^2 = x + 2$$

Step 2: Rearrange into standard form.

Move all terms to the left side:

$$x^2 - x - 2 = 0$$

Step 3: Check the discriminant.

$$\Delta = (-1)^2 - 4(1)(-2) = 1 + 8 = 9 > 0$$

Since $\Delta > 0$, there are two intersection points.

Step 4: Factor and solve.

We need two numbers that multiply to -2 and add to -1. Those are -2 and $+1$:

$$(x - 2)(x + 1) = 0$$

$$x = 2 \qquad \text{or} \qquad x = -1$$

Step 5: Find each y-value using $y = x + 2$.

For $x = 2$:

$$y = 2 + 2 = 4$$

For $x = -1$:

$$y = -1 + 2 = 1$$

Check: Point $(2, 4)$: $y = (2)^2 = 4$ ✓ and $y = 2 + 2 = 4$ ✓. Point $(-1, 1)$: $y = (-1)^2 = 1$ ✓ and $y = -1 + 2 = 1$ ✓.

$$\boxed{(2,\ 4) \text{ and } (-1,\ 1)}$$

Example 2 (Two Solutions, Downward Parabola): Solve the system.

$$\begin{cases} y = -x^2 + 4 \\ y = 2x + 1 \end{cases}$$

Step 1: Set the right-hand sides equal.

$$-x^2 + 4 = 2x + 1$$

Step 2: Rearrange into standard form.

Move all terms to the left side. Add x^2, subtract $2x$, subtract 1:

$$-x^2 + 4 - 2x - 1 = 0$$

$$-x^2 - 2x + 3 = 0$$

Multiply every term by -1 to make the leading coefficient positive:

$$x^2 + 2x - 3 = 0$$

Step 3: Check the discriminant.

$$\Delta = (2)^2 - 4(1)(-3) = 4 + 12 = 16 > 0$$

Step 4: Factor and solve.

Two numbers that multiply to -3 and add to $+2$: those are $+3$ and -1:

$$(x + 3)(x - 1) = 0$$

$$x = -3 \qquad \text{or} \qquad x = 1$$

Step 5: Find each y-value using $y = 2x + 1$.

For $x = -3$:

$$y = 2(-3) + 1 = -6 + 1 = -5$$

For $x = 1$:

$$y = 2(1) + 1 = 3$$

Check: Point $(-3, -5)$: $y = -(-3)^2 + 4 = -9 + 4 = -5$ ✓. Point $(1, 3)$: $y = -(1)^2 + 4 = 3$ ✓.

$$\boxed{(-3,\ -5) \text{ and } (1,\ 3)}$$

Example 3 (One Solution — Tangent Line): Solve the system.

$$\begin{cases} y = x^2 - 2x + 2 \\ y = 2x - 2 \end{cases}$$

Step 1: Set the right-hand sides equal.

$$x^2 - 2x + 2 = 2x - 2$$

Step 2: Rearrange into standard form.

$$x^2 - 2x + 2 - 2x + 2 = 0$$

$$x^2 - 4x + 4 = 0$$

Step 3: Check the discriminant.

$$\Delta = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

Since $\Delta = 0$, the line is tangent to the parabola. There is exactly one intersection point.

Step 4: Factor and solve.

The expression $x^2 - 4x + 4$ is a perfect square:

$$(x - 2)^2 = 0$$

$$x = 2$$

Step 5: Find y using $y = 2x - 2$.

$$y = 2(2) - 2 = 4 - 2 = 2$$

Check: $y = (2)^2 - 2(2) + 2 = 4 - 4 + 2 = 2$ ✓ and $y = 2(2) - 2 = 2$ ✓.

$$\boxed{(2,\ 2)}$$

Example 4 (No Solution): Solve the system.

$$\begin{cases} y = x^2 + 3 \\ y = x + 1 \end{cases}$$

Step 1: Set the right-hand sides equal.

$$x^2 + 3 = x + 1$$

Step 2: Rearrange into standard form.

$$x^2 - x + 2 = 0$$

Step 3: Check the discriminant.

$$\Delta = (-1)^2 - 4(1)(2) = 1 - 8 = -7$$

Since $\Delta < 0$, there are no real solutions. The line and parabola never intersect.

$$\boxed{\text{No solution}}$$

Example 5 (Two Solutions — Full Process): Solve the system.

$$\begin{cases} y = x^2 - x - 2 \\ y = 2x - 2 \end{cases}$$

Step 1: Set the right-hand sides equal.

$$x^2 - x - 2 = 2x - 2$$

Step 2: Rearrange into standard form.

Subtract $2x$ and add 2 to both sides:

$$x^2 - x - 2 - 2x + 2 = 0$$

$$x^2 - 3x = 0$$

Step 3: Check the discriminant.

$$\Delta = (-3)^2 - 4(1)(0) = 9 > 0$$

Step 4: Factor and solve.

Factor out x:

$$x(x - 3) = 0$$

$$x = 0 \qquad \text{or} \qquad x = 3$$

Step 5: Find each y-value using $y = 2x - 2$.

For $x = 0$:

$$y = 2(0) - 2 = -2$$

For $x = 3$:

$$y = 2(3) - 2 = 6 - 2 = 4$$

Check: Point $(0, -2)$: $y = 0 - 0 - 2 = -2$ ✓. Point $(3, 4)$: $y = 9 - 3 - 2 = 4$ ✓.

$$\boxed{(0,\ -2) \text{ and } (3,\ 4)}$$

Common Mistakes

- **Forgetting to move all terms to one side:** After setting the expressions equal, you must get 0 on one side before factoring. Trying to factor $x^2 - x = 2$ directly leads to errors.
- **Sign errors when rearranging:** When you move $2x + 1$ to the left, every term changes sign. Carefully subtract each term one at a time instead of doing it all in one step.
- **Substituting back into the quadratic instead of the linear:** Always substitute x back into the simpler (linear) equation to find y. The quadratic involves squaring, which is an extra step and an extra chance for error.
- **Only reporting one solution when there are two:** If the quadratic gives two x-values, you must find two ordered pairs. Stopping after the first one means your answer is incomplete.
- **Not checking the discriminant first:** Computing the discriminant takes five seconds and immediately tells you how many solutions to expect. If $\Delta < 0$, stop and write "no solution" — do not waste time trying to factor an unfactorable expression.

Worksheet

Solve each system. Write your answer as one or two ordered pairs, or state if there is no solution. Show all steps including the discriminant check.

1. $\begin{cases} y = x^2 \\ y = 2x + 3 \end{cases}$

2. $\begin{cases} y = x^2 - 4 \\ y = x + 2 \end{cases}$

3. $\begin{cases} y = x^2 - 1 \\ y = 2x + 2 \end{cases}$

4. $\begin{cases} y = -x^2 + 6 \\ y = x + 4 \end{cases}$

5. $\begin{cases} y = x^2 + 2x + 1 \\ y = 2x + 1 \end{cases}$

6. $\begin{cases} y = x^2 + 2 \\ y = x \end{cases}$

7. $\begin{cases} y = x^2 - 3x \\ y = 4 \end{cases}$

8. $\begin{cases} y = -x^2 + 9 \\ y = 2x + 1 \end{cases}$

9. *(Challenge)* $\begin{cases} y = x^2 - 2x \\ y = 3x - 6 \end{cases}$

10. *(Challenge)* $\begin{cases} y = x^2 - 2x - 3 \\ y = x + 1 \end{cases}$

Answer Key

1. $\begin{cases} y = x^2 \\ y = 2x + 3 \end{cases}$

Step 1: Set the right-hand sides equal.

$$x^2 = 2x + 3$$

Step 2: Rearrange into standard form.

$$x^2 - 2x - 3 = 0$$

Step 3: Discriminant check.

$$\Delta = (-2)^2 - 4(1)(-3) = 4 + 12 = 16 > 0 \quad \text{(two solutions)}$$

Step 4: Factor.

$$(x-3)(x+1) = 0 \implies x = 3 \quad \text{or} \quad x = -1$$

Step 5: Find y using $y = 2x + 3$.

$$x = 3: \quad y = 2(3) + 3 = 9$$

$$x = -1: \quad y = 2(-1) + 3 = 1$$

$$\boxed{(3,\ 9) \text{ and } (-1,\ 1)}$$

2. $\begin{cases} y = x^2 - 4 \\ y = x + 2 \end{cases}$

Step 1: Set equal.

$$x^2 - 4 = x + 2$$

Step 2: Rearrange.

$$x^2 - x - 6 = 0$$

Step 3: Discriminant check.

$$\Delta = (-1)^2 - 4(1)(-6) = 1 + 24 = 25 > 0$$

Step 4: Factor.

$$(x-3)(x+2) = 0 \implies x = 3 \quad \text{or} \quad x = -2$$

Step 5: Find y using $y = x + 2$.

$$x = 3: \quad y = 3 + 2 = 5$$

$$x = -2: \quad y = -2 + 2 = 0$$

$$\boxed{(3,\ 5) \text{ and } (-2,\ 0)}$$

3. $\begin{cases} y = x^2 - 1 \\ y = 2x + 2 \end{cases}$

Step 1: Set equal.

$$x^2 - 1 = 2x + 2$$

Step 2: Rearrange.

$$x^2 - 2x - 3 = 0$$

Step 3: Discriminant check.

$$\Delta = (-2)^2 - 4(1)(-3) = 4 + 12 = 16 > 0$$

Step 4: Factor.

$$(x-3)(x+1) = 0 \implies x = 3 \quad \text{or} \quad x = -1$$

Step 5: Find y using $y = 2x + 2$.

$$x = 3: \quad y = 2(3) + 2 = 8$$

$$x = -1: \quad y = 2(-1) + 2 = 0$$

$$\boxed{(3,\ 8) \text{ and } (-1,\ 0)}$$

4. $\begin{cases} y = -x^2 + 6 \\ y = x + 4 \end{cases}$

Step 1: Set equal.

$$-x^2 + 6 = x + 4$$

Step 2: Rearrange. Move everything to the left and multiply by -1:

$$-x^2 - x + 2 = 0$$

$$x^2 + x - 2 = 0$$

Step 3: Discriminant check.

$$\Delta = (1)^2 - 4(1)(-2) = 1 + 8 = 9 > 0$$

Step 4: Factor.

$$(x + 2)(x - 1) = 0 \implies x = -2 \quad \text{or} \quad x = 1$$

Step 5: Find y using $y = x + 4$.

$$x = -2: \quad y = -2 + 4 = 2$$

$$x = 1: \quad y = 1 + 4 = 5$$

$$\boxed{(-2,\ 2) \text{ and } (1,\ 5)}$$

5. $\begin{cases} y = x^2 + 2x + 1 \\ y = 2x + 1 \end{cases}$

Step 1: Set equal.

$$x^2 + 2x + 1 = 2x + 1$$

Step 2: Rearrange. Subtract $2x + 1$ from both sides:

$$x^2 = 0$$

Step 3: Discriminant check.

The equation is already $x^2 = 0$, which means $a = 1$, $b = 0$, $c = 0$:

$$\Delta = (0)^2 - 4(1)(0) = 0 \quad \text{(tangent line — one solution)}$$

Step 4: Solve.

$$x = 0$$

Step 5: Find y using $y = 2x + 1$.

$$y = 2(0) + 1 = 1$$

$$\boxed{(0,\ 1)}$$

6. $\begin{cases} y = x^2 + 2 \\ y = x \end{cases}$

Step 1: Set equal.

$$x^2 + 2 = x$$

Step 2: Rearrange.

$$x^2 - x + 2 = 0$$

Step 3: Discriminant check.

$$\Delta = (-1)^2 - 4(1)(2) = 1 - 8 = -7 < 0 \quad \text{(no solution)}$$

Since $\Delta < 0$, the line and parabola never intersect.

$$\boxed{\text{No solution}}$$

7. $\begin{cases} y = x^2 - 3x \\ y = 4 \end{cases}$

Step 1: Set equal.

$$x^2 - 3x = 4$$

Step 2: Rearrange.

$$x^2 - 3x - 4 = 0$$

Step 3: Discriminant check.

$$\Delta = (-3)^2 - 4(1)(-4) = 9 + 16 = 25 > 0$$

Step 4: Factor.

$$(x - 4)(x + 1) = 0 \implies x = 4 \quad \text{or} \quad x = -1$$

Step 5: Find y. The linear equation is $y = 4$, so both points have $y = 4$.

$$x = 4: \quad y = 4$$

$$x = -1: \quad y = 4$$

$$\boxed{(4,\ 4) \text{ and } (-1,\ 4)}$$

8. $\begin{cases} y = -x^2 + 9 \\ y = 2x + 1 \end{cases}$

Step 1: Set equal.

$$-x^2 + 9 = 2x + 1$$

Step 2: Rearrange. Move all terms left, then multiply by -1:

$$-x^2 - 2x + 8 = 0$$

$$x^2 + 2x - 8 = 0$$

Step 3: Discriminant check.

$$\Delta = (2)^2 - 4(1)(-8) = 4 + 32 = 36 > 0$$

Step 4: Factor.

$$(x + 4)(x - 2) = 0 \implies x = -4 \quad \text{or} \quad x = 2$$

Step 5: Find y using $y = 2x + 1$.

$$x = -4: \quad y = 2(-4) + 1 = -8 + 1 = -7$$

$$x = 2: \quad y = 2(2) + 1 = 5$$

$$\boxed{(-4,\ -7) \text{ and } (2,\ 5)}$$

9. *(Challenge)* $\begin{cases} y = x^2 - 2x \\ y = 3x - 6 \end{cases}$

Step 1: Set equal.

$$x^2 - 2x = 3x - 6$$

Step 2: Rearrange.

$$x^2 - 2x - 3x + 6 = 0$$
$$x^2 - 5x + 6 = 0$$

Step 3: Discriminant check.

$$\Delta = (-5)^2 - 4(1)(6) = 25 - 24 = 1 > 0$$

Step 4: Factor.

Two numbers that multiply to $+6$ and add to -5: those are -2 and -3:

$$(x-2)(x-3)=0 \implies x=2 \quad \text{or} \quad x=3$$

Step 5: Find y using $y=3x-6$.

$$x=2: \quad y=3(2)-6=0$$
$$x=3: \quad y=3(3)-6=3$$

$$\boxed{(2,\ 0) \text{ and } (3,\ 3)}$$

10. *(Challenge)* $\begin{cases} y=x^2-2x-3 \\ y=x+1 \end{cases}$

Step 1: Set equal.

$$x^2-2x-3=x+1$$

Step 2: Rearrange.

$$x^2-2x-3-x-1=0$$
$$x^2-3x-4=0$$

Step 3: Discriminant check.

$$\Delta=(-3)^2-4(1)(-4)=9+16=25>0$$

Step 4: Factor.

Two numbers that multiply to -4 and add to -3: those are -4 and $+1$:

$$(x-4)(x+1)=0 \implies x=4 \quad \text{or} \quad x=-1$$

Step 5: Find y using $y=x+1$.

$$x=4: \quad y=4+1=5$$
$$x=-1: \quad y=-1+1=0$$

$$\boxed{(4,\ 5) \text{ and } (-1,\ 0)}$$

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End of Lesson · ApexMath
