Quadratic-Quadratic Systems
ApexMath

Notes

What Is a Quadratic-Quadratic System?

A **quadratic-quadratic system** is a system where *both* equations are parabolas. We are looking for every point (x, y) where the two parabolas **intersect** — where both equations are true at the same time.

Three Possible Outcomes

Two parabolas can relate to each other in three ways.

- **Two solutions:** The parabolas cross at two distinct points.
- **One solution:** The parabolas touch at exactly one point. This happens when one parabola is tangent to the other.
- **No solution:** The parabolas never meet. One sits entirely above or inside the other.

The Key Technique: Subtract the Equations

With two quadratic equations, the most powerful first move is to **subtract one equation from the other**. When both equations are in the form $y = \ldots$, subtracting eliminates y and — very often — also eliminates the x^2 term, leaving a much simpler equation to solve.

There are two situations:

- **Same leading coefficient** (e.g. both have x^2): Subtracting removes the x^2 term entirely and leaves a *linear* equation. Solve it for x, then substitute back. This gives at most one solution.
- **Different leading coefficients** (e.g. x^2 and $-x^2$): Subtracting leaves a *quadratic* equation with no y. Solve it for x (usually by factoring), then substitute each x back. This can give two solutions.

The Four Steps

Step 1: Write both equations in the form $y = \ldots$. Label them Equation 1 and Equation 2.

Step 2: Subtract one equation from the other to eliminate y. Rearrange the result into a solvable form.

Step 3: Solve for x. If the result is linear, solve directly. If it is quadratic, use the discriminant, then factor or the quadratic formula.

Step 4: Substitute each x-value back into either original equation to find the corresponding y-value. Write each answer as an ordered pair (x, y).

Always Check Your Answer

Substitute each solution into *both* original equations to confirm.

Pro Tip

- Before subtracting, look at the leading coefficients of both equations. If they are the same, you will get a linear equation after subtracting — a strong hint that there is at most one solution. If they are different, you will get a quadratic and likely two solutions.
- Always subtract the *entire* equation — including all terms on the right-hand side. Write out the subtraction line by line rather than trying to do it mentally; sign errors here are the most common mistake in this lesson.
- After subtracting, double-check the signs on every term. A misplaced negative sign will give you the wrong x-values for the rest of the problem.
- When substituting x back in, use the *simpler* of the two original equations. Choose the one with fewer terms or smaller coefficients.
- If the subtraction leaves $0 = k$ where $k \neq 0$, there is no solution. If it leaves $0 = 0$, the two parabolas are identical and there are infinitely many solutions.

Examples

Example 1 (Two Solutions — Opposite Signs): Solve the system.

$$\begin{cases} y = x^2 + 2 \\ y = -x^2 + 4 \end{cases}$$

Step 1: Both equations are already in $y = \ldots$ form.

Step 2: Subtract Equation 2 from Equation 1 to eliminate y.

Write it out term by term:

$$(y) - (y) = (x^2 + 2) - (-x^2 + 4)$$

$$0 = x^2 + 2 + x^2 - 4$$

$$0 = 2x^2 - 2$$

Step 3: Solve for x.

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = 1 \qquad \text{or} \qquad x = -1$$

Step 4: Substitute each x into Equation 1 to find y.

For $x = 1$:

$$y = (1)^2 + 2 = 1 + 2 = 3$$

For $x = -1$:

$$y = (-1)^2 + 2 = 1 + 2 = 3$$

Check: Point $(1, 3)$: eq1 $3 = 1 + 2$ ✓ eq2 $3 = -1 + 4$ ✓. Point $(-1, 3)$: eq1 $3 = 1 + 2$ ✓ eq2 $3 = -1 + 4$ ✓.

$$\boxed{(1,\ 3) \text{ and } (-1,\ 3)}$$

Example 2 (Two Solutions — Different Leading Coefficients): Solve the system.

$$\begin{cases} y = x^2 \\ y = -x^2 + 8 \end{cases}$$

Step 2: Subtract Equation 2 from Equation 1.

$$0 = x^2 - (-x^2 + 8)$$

$$0 = x^2 + x^2 - 8$$

$$0 = 2x^2 - 8$$

Step 3: Solve for x.

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = 2 \qquad \text{or} \qquad x = -2$$

Step 4: Substitute each x into $y = x^2$.

For $x = 2$:

$$y = (2)^2 = 4$$

For $x = -2$:

$$y = (-2)^2 = 4$$

Check: Point $(2, 4)$: eq1 $4 = 4$ ✓ eq2 $4 = -4 + 8 = 4$ ✓. Point $(-2, 4)$: eq1 $4 = 4$ ✓ eq2 $4 = -4 + 8 = 4$ ✓.

$$\boxed{(2,\ 4) \text{ and } (-2,\ 4)}$$

Example 3 (One Solution — Same Leading Coefficient): Solve the system.

$$\begin{cases} y = x^2 + 2x \\ y = x^2 - 2x + 8 \end{cases}$$

Step 2: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + 2x) - (x^2 - 2x + 8)$$

Distribute the subtraction carefully across every term:

$$0 = x^2 + 2x - x^2 + 2x - 8$$

$$0 = 4x - 8$$

The x^2 terms cancelled. We now have a linear equation.

Step 3: Solve for x.

$$4x = 8$$

$$x = 2$$

Step 4: Substitute $x = 2$ into Equation 1.

$$y = (2)^2 + 2(2) = 4 + 4 = 8$$

Check: eq1: $8 = 4 + 4$ ✓ eq2: $8 = 4 - 4 + 8$ ✓.

$$\boxed{(2,\ 8)}$$

Example 4 (No Solution): Solve the system.

$$\begin{cases} y = x^2 + 3 \\ y = -x^2 + 1 \end{cases}$$

Step 2: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + 3) - (-x^2 + 1)$$

$$0 = x^2 + 3 + x^2 - 1$$

$$0 = 2x^2 + 2$$

Step 3: Try to solve for x.

$$2x^2 = -2$$

$$x^2 = -1$$

There is no real number whose square is -1. The parabolas never intersect.

$$\boxed{\text{No solution}}$$

Example 5 (Two Solutions — Verify Carefully): Solve the system.

$$\begin{cases} y = x^2 - 4 \\ y = -x^2 + 4 \end{cases}$$

Step 2: Subtract Equation 2 from Equation 1.

$$0 = (x^2 - 4) - (-x^2 + 4)$$

$$0 = x^2 - 4 + x^2 - 4$$

$$0 = 2x^2 - 8$$

Step 3: Solve for x.

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = 2 \qquad \text{or} \qquad x = -2$$

Step 4: Substitute each x into $y = x^2 - 4$.

For $x = 2$:

$$y = (2)^2 - 4 = 4 - 4 = 0$$

For $x = -2$:

$$y = (-2)^2 - 4 = 4 - 4 = 0$$

Check: Point $(2, 0)$: eq1 $0 = 4 - 4$ ✓ eq2 $0 = -4 + 4$ ✓. Point $(-2, 0)$: eq1 $0 = 4 - 4$ ✓ eq2 $0 = -4 + 4$ ✓.

$$\boxed{(2,\ 0) \text{ and } (-2,\ 0)}$$

Common Mistakes

- **Sign errors when subtracting:** When subtracting Equation 2, every term in Equation 2 changes sign. Students frequently forget to flip the sign on the constant term. Write the subtraction line by line and change each sign individually.
- **Subtracting instead of the whole equation:** The subtraction must apply to every single term — including the right-hand side constant. Do not subtract only part of the equation.
- **Stopping when x^2 = negative number:** This means there is no real solution. Do not try to take the square root of a negative number. Simply state "No solution" and move on.
- **Forgetting to find y for both x-values:** When subtraction gives two values of x, you need two ordered pairs. Each x must be substituted back separately.
- **Substituting into the wrong equation:** After solving for x, substitute into one of the *original* equations — not the subtracted result, which no longer contains y.

Worksheet

Solve each system. Write your answer as one or two ordered pairs, or state if there is no solution. Show the subtraction step clearly.

1. $\begin{cases} y = x^2 \\ y = -x^2 + 2 \end{cases}$

2. $\begin{cases} y = x^2 + 3x \\ y = x^2 - x + 8 \end{cases}$

3. $\begin{cases} y = x^2 - 3 \\ y = -x^2 + 5 \end{cases}$

4. $\begin{cases} y = 2x^2 \\ y = x^2 + 4 \end{cases}$

5. $\begin{cases} y = x^2 - 2x + 3 \\ y = x^2 + 2x - 1 \end{cases}$

6. $\begin{cases} y = x^2 + 4 \\ y = -x^2 + 2 \end{cases}$

7. $\begin{cases} y = x^2 - 5 \\ y = -x^2 + 3 \end{cases}$

8. $\begin{cases} y = 3x^2 \\ y = x^2 + 8 \end{cases}$

9. *(Challenge)* $\begin{cases} y = x^2 + 2x - 3 \\ y = -x^2 + 2x + 5 \end{cases}$

10. *(Challenge)* $\begin{cases} y = x^2 + x - 2 \\ y = -x^2 - x + 10 \end{cases}$

Answer Key

1. $\begin{cases} y = x^2 \\ y = -x^2 + 2 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = x^2 - (-x^2 + 2) = x^2 + x^2 - 2 = 2x^2 - 2$$

Step 2: Solve for x.

$$2x^2 = 2 \implies x^2 = 1 \implies x = 1 \quad \text{or} \quad x = -1$$

Step 3: Substitute into $y = x^2$.

$$x = 1: \quad y = 1 \qquad x = -1: \quad y = 1$$

$$\boxed{(1,\ 1) \text{ and } (-1,\ 1)}$$

2. $\begin{cases} y = x^2 + 3x \\ y = x^2 - x + 8 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + 3x) - (x^2 - x + 8)$$
$$0 = x^2 + 3x - x^2 + x - 8$$
$$0 = 4x - 8$$

The x^2 terms cancelled, giving a linear equation.

Step 2: Solve for x.

$$4x = 8 \implies x = 2$$

Step 3: Substitute $x = 2$ into Equation 1.

$$y = (2)^2 + 3(2) = 4 + 6 = 10$$

$$\boxed{(2,\ 10)}$$

3. $\begin{cases} y = x^2 - 3 \\ y = -x^2 + 5 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 - 3) - (-x^2 + 5) = x^2 - 3 + x^2 - 5 = 2x^2 - 8$$

Step 2: Solve for x.

$$2x^2 = 8 \implies x^2 = 4 \implies x = 2 \quad \text{or} \quad x = -2$$

Step 3: Substitute into $y = x^2 - 3$.

$$x = 2: \quad y = 4 - 3 = 1 \qquad x = -2: \quad y = 4 - 3 = 1$$

$$\boxed{(2,\ 1) \text{ and } (-2,\ 1)}$$

4. $\begin{cases} y = 2x^2 \\ y = x^2 + 4 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = 2x^2 - (x^2 + 4) = 2x^2 - x^2 - 4 = x^2 - 4$$

Step 2: Solve for x.

$$x^2 = 4 \implies x = 2 \quad \text{or} \quad x = -2$$

Step 3: Substitute into $y = 2x^2$.

$$x = 2: \quad y = 2(4) = 8 \qquad x = -2: \quad y = 2(4) = 8$$

$$\boxed{(2,\ 8) \text{ and } (-2,\ 8)}$$

5. $\begin{cases} y = x^2 - 2x + 3 \\ y = x^2 + 2x - 1 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 - 2x + 3) - (x^2 + 2x - 1)$$
$$0 = x^2 - 2x + 3 - x^2 - 2x + 1$$
$$0 = -4x + 4$$

Step 2: Solve for x.

$$4x = 4 \implies x = 1$$

Step 3: Substitute $x = 1$ into Equation 1.

$$y = (1)^2 - 2(1) + 3 = 1 - 2 + 3 = 2$$

$$\boxed{(1,\ 2)}$$

6. $\begin{cases} y = x^2 + 4 \\ y = -x^2 + 2 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + 4) - (-x^2 + 2) = x^2 + 4 + x^2 - 2 = 2x^2 + 2$$

Step 2: Try to solve for x.

$$2x^2 = -2 \implies x^2 = -1$$

No real number squares to -1.

$$\boxed{\text{No solution}}$$

7. $\begin{cases} y = x^2 - 5 \\ y = -x^2 + 3 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 - 5) - (-x^2 + 3) = x^2 - 5 + x^2 - 3 = 2x^2 - 8$$

Step 2: Solve for x.

$$2x^2 = 8 \implies x^2 = 4 \implies x = 2 \quad \text{or} \quad x = -2$$

Step 3: Substitute into $y = x^2 - 5$.

$$x = 2: \quad y = 4 - 5 = -1 \qquad x = -2: \quad y = 4 - 5 = -1$$

$$\boxed{(2,\ -1) \text{ and } (-2,\ -1)}$$

8. $\begin{cases} y = 3x^2 \\ y = x^2 + 8 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = 3x^2 - (x^2 + 8) = 3x^2 - x^2 - 8 = 2x^2 - 8$$

Step 2: Solve for x.

$$2x^2 = 8 \implies x^2 = 4 \implies x = 2 \quad \text{or} \quad x = -2$$

Step 3: Substitute into $y = 3x^2$.

$$x = 2: \quad y = 3(4) = 12 \qquad x = -2: \quad y = 3(4) = 12$$

$$\boxed{(2,\ 12) \text{ and } (-2,\ 12)}$$

9. *(Challenge)* $\begin{cases} y = x^2 + 2x - 3 \\ y = -x^2 + 2x + 5 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + 2x - 3) - (-x^2 + 2x + 5)$$

Distribute the subtraction carefully across every term:

$$0 = x^2 + 2x - 3 + x^2 - 2x - 5$$

$$0 = 2x^2 - 8$$

Step 2: Solve for x.

$$2x^2 = 8 \implies x^2 = 4 \implies x = 2 \quad \text{or} \quad x = -2$$

Step 3: Substitute into Equation 1.

For $x = 2$:

$$y = (2)^2 + 2(2) - 3 = 4 + 4 - 3 = 5$$

For $x = -2$:

$$y = (-2)^2 + 2(-2) - 3 = 4 - 4 - 3 = -3$$

$$\boxed{(2,\ 5) \text{ and } (-2,\ -3)}$$

10. *(Challenge)* $\begin{cases} y = x^2 + x - 2 \\ y = -x^2 - x + 10 \end{cases}$

Step 1: Subtract Equation 2 from Equation 1.

$$0 = (x^2 + x - 2) - (-x^2 - x + 10)$$

$$0 = x^2 + x - 2 + x^2 + x - 10$$

$$0 = 2x^2 + 2x - 12$$

Divide every term by 2 to simplify:

$$0 = x^2 + x - 6$$

Step 2: Factor and solve.

Two numbers that multiply to -6 and add to $+1$: those are $+3$ and -2:

$$(x + 3)(x - 2) = 0 \implies x = -3 \quad \text{or} \quad x = 2$$

Step 3: Substitute into Equation 1.

For $x = 2$:

$$y = (2)^2 + 2 - 2 = 4 + 2 - 2 = 4$$

For $x = -3$:

$$y = (-3)^2 + (-3) - 2 = 9 - 3 - 2 = 4$$

$$\boxed{(2,\ 4) \text{ and } (-3,\ 4)}$$

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