Solving Linear Systems by Elimination
ApexMath

Notes

What Is Elimination?

Elimination is a second method for solving a system of two linear equations. Instead of isolating a variable and substituting, we **add the two equations together** in a way that makes one variable disappear entirely. Once one variable is gone, we solve for the other, then substitute back to find the first.

Elimination is often faster than substitution when neither equation has a variable with a coefficient of 1.

The Core Idea

If two equations contain the term $+3y$ and $-3y$, adding them causes those terms to cancel:

$$(5x + 3y) + (2x - 3y) = 7x + 0y = 7x$$

That cancellation is the entire point. We arrange for the coefficients of one variable to be **opposites** (same number, opposite signs), then add.

The Four Steps of Elimination

Step 1: Look at the coefficients of each variable across both equations. Decide which variable to eliminate.

Step 2: If the coefficients are not already opposites, multiply one or both equations by constants so that they become opposites.

Step 3: Add the two equations. One variable will cancel. Solve for the remaining variable.

Step 4: Substitute the value you found back into either original equation and solve for the second variable. Write your answer as an ordered pair (x, y).

Three Cases to Know

- **One solution:** The variable cancels and you are left with a single equation in one unknown. Solve it normally.
- **No solution:** Both variables cancel and you are left with a statement that is *never* true (e.g. $0 = 5$). The lines are parallel.
- **Infinitely many solutions:** Both variables cancel and you are left with a statement that is *always* true (e.g. $0 = 0$). The two equations describe the same line.

Always Check Your Answer

Substitute both values into *both* original equations to confirm. This catches arithmetic errors before they cost you marks.

Pro Tip

- Before multiplying, scan both equations for a variable whose coefficients are already equal or already opposites. If they are already opposite, you can add immediately — no multiplication needed.
- When you multiply an equation by a constant, every single term on both sides must be multiplied. Forgetting to multiply the right-hand side is one of the most common errors in this method.
- You can choose to eliminate either variable — pick whichever requires the smallest numbers to work with. Smaller multipliers mean fewer chances for sign errors.
- After adding the equations, double-check that the variable you intended to eliminate is actually gone. If it is still there, the coefficients were not true opposites and you need to redo Step 2.
- Substitution and elimination always give the same answer. If the problem looks messy with elimination, switch to substitution — and vice versa.

Examples

Example 1 (Direct Cancellation): Solve the system.

$$\begin{cases} 4x + 3y = 10 \\ 2x - 3y = 2 \end{cases}$$

Step 1: Identify which variable to eliminate.

Look at the y-coefficients: $+3y$ and $-3y$. These are already opposites. No multiplication is needed.

Step 2: Add the two equations.

$$(4x + 3y) + (2x - 3y) = 10 + 2$$

$$6x + 0y = 12$$
$$6x = 12$$

Step 3: Solve for x.

$$x = 2$$

Step 4: Substitute $x = 2$ into the first equation and solve for y.

$$4(2) + 3y = 10$$

$$8 + 3y = 10$$
$$3y = 2$$

$$y = \frac{2}{3}$$

Check: Equation 1: $4(2) + 3\left(\frac{2}{3}\right) = 8 + 2 = 10$ ✓ Equation 2: $2(2) - 3\left(\frac{2}{3}\right) = 4 - 2 = 2$ ✓

$$\boxed{\left(2,\ \frac{2}{3}\right)}$$

Example 2 (Multiply One Equation): Solve the system.

$$\begin{cases} 3x + y = 7 \\ x + 4y = 9 \end{cases}$$

Step 1: Decide which variable to eliminate.

Look at the y-coefficients: $+1$ and $+4$. To make them opposites, multiply the first equation by 4 and the second by -1. Or, more simply: multiply the first equation by -4 so the y-terms become $-4y$ and $+4y$.

Step 2: Multiply the first equation by -4.

$$-4(3x + y) = -4(7) \quad \Longrightarrow \quad -12x - 4y = -28$$

The second equation stays as-is:

$$x + 4y = 9$$

Step 3: Add the two equations.

$$(-12x - 4y) + (x + 4y) = -28 + 9$$

$$-11x + 0y = -19$$

$$-11x = -19$$

$$x = \frac{19}{11}$$

Step 4: Substitute $x = \frac{19}{11}$ into the first original equation.

$$3\left(\frac{19}{11}\right) + y = 7$$

$$\frac{57}{11} + y = \frac{77}{11}$$

$$y = \frac{77}{11} - \frac{57}{11} = \frac{20}{11}$$

Check: Equation 1: $3\left(\frac{19}{11}\right)+\frac{20}{11}=\frac{57}{11}+\frac{20}{11}=\frac{77}{11}=7$ ✓ Equation 2: $\frac{19}{11}+4\left(\frac{20}{11}\right)=\frac{19}{11}+\frac{80}{11}=\frac{99}{11}=9$ ✓

$$\boxed{\left(\frac{19}{11},\ \frac{20}{11}\right)}$$

Example 3 (Multiply Both Equations): Solve the system.

$$\begin{cases} 2x+3y=12 \\ 3x+2y=13 \end{cases}$$

Step 1: Decide which variable to eliminate.

Neither the x-coefficients (2 and 3) nor the y-coefficients (3 and 2) are opposites. We need to multiply both equations. To eliminate y, make the coefficients of y into $+6$ and -6.

Step 2: Multiply the first equation by 2 and the second by -3.

$$2(2x+3y)=2(12) \quad\Longrightarrow\quad 4x+6y=24$$

$$-3(3x+2y)=-3(13) \quad\Longrightarrow\quad -9x-6y=-39$$

Step 3: Add the two new equations.

$$(4x+6y)+(-9x-6y)=24+(-39)$$

$$-5x+0y=-15$$

$$-5x=-15$$

$$x=3$$

Step 4: Substitute $x=3$ into the first original equation.

$$2(3)+3y=12$$

$$6+3y=12$$

$$3y=6$$

$$y=2$$

Check: Equation 1: $2(3)+3(2)=6+6=12$ ✓ Equation 2: $3(3)+2(2)=9+4=13$ ✓

$$\boxed{(3,\ 2)}$$

Example 4 (No Solution): Solve the system.

$$\begin{cases} 3x-6y=9 \\ x-2y=5 \end{cases}$$

Step 1: Decide which variable to eliminate.

Multiply the second equation by -3 to make the x-coefficients opposites ($+3$ and -3).

Step 2: Multiply the second equation by -3.

$$-3(x - 2y) = -3(5) \quad \Longrightarrow \quad -3x + 6y = -15$$

Step 3: Add to the first equation.

$$(3x - 6y) + (-3x + 6y) = 9 + (-15)$$
$$0 = -6$$

This is **never true**. The lines are parallel and never intersect.

$$\boxed{\text{No solution}}$$

Example 5 (Infinitely Many Solutions): Solve the system.

$$\begin{cases} 4x + 2y = 10 \\ 2x + y = 5 \end{cases}$$

Step 1: Multiply the second equation by -2.

$$-2(2x + y) = -2(5) \quad \Longrightarrow \quad -4x - 2y = -10$$

Step 2: Add to the first equation.

$$(4x + 2y) + (-4x - 2y) = 10 + (-10)$$
$$0 = 0$$

This is **always true**. The two equations represent the same line. Every point on that line is a solution.

$$\boxed{\text{Infinitely many solutions}}$$

Common Mistakes

- **Only multiplying part of an equation:** When you multiply an equation by a constant, every term — including the right-hand side — must be multiplied. Forgetting the right-hand side is the single most frequent error in this method.
- **Adding instead of subtracting (or vice versa):** The goal is to get opposite coefficients so the terms cancel when added. If the coefficients are equal (not opposite), you need to subtract one equation from the other, not add. Check the signs carefully.

- **Choosing the wrong equation to substitute back into:** After finding one variable, you can substitute into *either* original equation — but it must be one of the original equations, not the scaled version you created in Step 2.
- **Stopping after finding one variable:** As always, one number is not a complete answer. You must find both x and y and write them as an ordered pair (x, y).
- **Misreading the special cases:** $0 = 0$ means infinitely many solutions. $0 = k$ where $k \neq 0$ means no solution. Students often confuse these two outcomes.

Worksheet

Solve each system using the elimination method. Write your answer as an ordered pair (x, y), or state if there is no solution or infinitely many solutions.

1. $\begin{cases} 3x + y = 10 \\ x - y = 2 \end{cases}$

2. $\begin{cases} 2x + 3y = 12 \\ -2x + y = 4 \end{cases}$

3. $\begin{cases} x + 2y = 7 \\ 3x - y = 7 \end{cases}$

4. $\begin{cases} 4x + 3y = 11 \\ 2x - y = 1 \end{cases}$

5. $\begin{cases} 3x + 2y = 5 \\ 2x + 3y = 5 \end{cases}$

6. $\begin{cases} 5x + 2y = 8 \\ 3x + 4y = 10 \end{cases}$

7. $\begin{cases} 2x - 4y = 6 \\ x - 2y = 5 \end{cases}$

8. $\begin{cases} 6x + 4y = 8 \\ 3x + 2y = 4 \end{cases}$

9. *(Challenge)* $\begin{cases} 2x + 5y = 16 \\ 3x - 2y = 5 \end{cases}$

10. *(Challenge)* $\begin{cases} 4x + 3y = 3 \\ 3x - 5y = 24 \end{cases}$

Answer Key

1. $\begin{cases} 3x + y = 10 \\ x - y = 2 \end{cases}$

Step 1: The y-coefficients are $+1$ and -1. They are already opposites. No multiplication needed.

Step 2: Add the two equations.

$$(3x + y) + (x - y) = 10 + 2$$

$$4x = 12$$

$$x = 3$$

Step 3: Substitute $x = 3$ into the first equation.

$$3(3) + y = 10$$

$$9 + y = 10$$

$$y = 1$$

$$\boxed{(3,\ 1)}$$

2. $\begin{cases} 2x + 3y = 12 \\ -2x + y = 4 \end{cases}$

Step 1: The x-coefficients are $+2$ and -2. They are already opposites.

Step 2: Add the two equations.

$$(2x + 3y) + (-2x + y) = 12 + 4$$

$$4y = 16$$

$$y = 4$$

Step 3: Substitute $y = 4$ into the first equation.

$$2x + 3(4) = 12$$

$$2x + 12 = 12$$

$$2x = 0$$

$$x = 0$$

$$\boxed{(0,\ 4)}$$

3. $\begin{cases} x + 2y = 7 \\ 3x - y = 7 \end{cases}$

Step 1: The y-coefficients are $+2$ and -1. Multiply the second equation by 2 to make them $+2$ and -2.

Step 2: Multiply the second equation by 2.

$$2(3x - y) = 2(7) \quad \Longrightarrow \quad 6x - 2y = 14$$

Step 3: Add to the first equation.

$$(x + 2y) + (6x - 2y) = 7 + 14$$

$$7x = 21$$

$$x = 3$$

Step 4: Substitute $x = 3$ into the first equation.

$$3 + 2y = 7$$

$$2y = 4$$

$$y = 2$$

$$\boxed{(3,\ 2)}$$

4. $\begin{cases} 4x + 3y = 11 \\ 2x - y = 1 \end{cases}$

Step 1: The y-coefficients are $+3$ and -1. Multiply the second equation by 3 to make them $+3$ and -3.

Step 2: Multiply the second equation by 3.

$$3(2x - y) = 3(1) \quad \Longrightarrow \quad 6x - 3y = 3$$

Step 3: Add to the first equation.

$$(4x + 3y) + (6x - 3y) = 11 + 3$$

$$10x = 14$$

$$x = \frac{14}{10} = \frac{7}{5}$$

Step 4: Substitute $x = \frac{7}{5}$ into the second original equation.

$$2\left(\frac{7}{5}\right) - y = 1$$

$$\frac{14}{5} - y = \frac{5}{5}$$

$$y = \frac{14}{5} - \frac{5}{5} = \frac{9}{5}$$

$$\boxed{\left(\frac{7}{5}, \frac{9}{5}\right)}$$

5. $\begin{cases} 3x + 2y = 5 \\ 2x + 3y = 5 \end{cases}$

Step 1: No coefficients are already opposites. To eliminate y, make the y-coefficients $+6$ and -6: multiply the first equation by 3 and the second equation by -2.

Step 2: Multiply.

$$3(3x + 2y) = 3(5) \quad \Longrightarrow \quad 9x + 6y = 15$$
$$-2(2x + 3y) = -2(5) \quad \Longrightarrow \quad -4x - 6y = -10$$

Step 3: Add the two new equations.

$$(9x + 6y) + (-4x - 6y) = 15 + (-10)$$
$$5x = 5$$
$$x = 1$$

Step 4: Substitute $x = 1$ into the first original equation.

$$3(1) + 2y = 5$$
$$2y = 2$$
$$y = 1$$

$$\boxed{(1,\ 1)}$$

6. $\begin{cases} 5x + 2y = 8 \\ 3x + 4y = 10 \end{cases}$

Step 1: Multiply the first equation by 2 to make the y-coefficients $+4$ and $+4$. Since they will be equal (not opposite), we subtract the second equation instead of adding.

Step 2: Multiply the first equation by 2.

$$2(5x + 2y) = 2(8) \quad \Longrightarrow \quad 10x + 4y = 16$$

Step 3: Subtract the second original equation from this result.

$$(10x + 4y) - (3x + 4y) = 16 - 10$$
$$7x = 6$$
$$x = \frac{6}{7}$$

Step 4: Substitute $x = \dfrac{6}{7}$ into the first original equation.

$$5\left(\frac{6}{7}\right) + 2y = 8$$

$$\frac{30}{7} + 2y = \frac{56}{7}$$

$$2y = \frac{56}{7} - \frac{30}{7} = \frac{26}{7}$$

$$y = \frac{13}{7}$$

$$\boxed{\left(\frac{6}{7}, \frac{13}{7}\right)}$$

7. $\begin{cases} 2x - 4y = 6 \\ x - 2y = 5 \end{cases}$

Step 1: Multiply the second equation by -2 to make the x-coefficients $+2$ and -2.

Step 2: Multiply.

$$-2(x - 2y) = -2(5) \quad \Longrightarrow \quad -2x + 4y = -10$$

Step 3: Add to the first equation.

$$(2x - 4y) + (-2x + 4y) = 6 + (-10)$$

$$0 = -4$$

This is never true. The lines are parallel.

$$\boxed{\text{No solution}}$$

8. $\begin{cases} 6x + 4y = 8 \\ 3x + 2y = 4 \end{cases}$

Step 1: Multiply the second equation by -2 to make the x-coefficients $+6$ and -6.

Step 2: Multiply.

$$-2(3x + 2y) = -2(4) \quad \Longrightarrow \quad -6x - 4y = -8$$

Step 3: Add to the first equation.

$$(6x + 4y) + (-6x - 4y) = 8 + (-8)$$

$$0 = 0$$

This is always true. Both equations represent the same line.

$$\boxed{\text{Infinitely many solutions}}$$

9. *(Challenge)* $\begin{cases} 2x + 5y = 16 \\ 3x - 2y = 5 \end{cases}$

Step 1: No coefficients are already opposites. To eliminate y, make the y-coefficients $+10$ and -10: multiply the first equation by 2 and the second equation by 5.

Step 2: Multiply.

$$2(2x + 5y) = 2(16) \quad \Longrightarrow \quad 4x + 10y = 32$$

$$5(3x - 2y) = 5(5) \quad \Longrightarrow \quad 15x - 10y = 25$$

Step 3: Add the two new equations.

$$(4x + 10y) + (15x - 10y) = 32 + 25$$

$$19x = 57$$

$$x = 3$$

Step 4: Substitute $x = 3$ into the first original equation.

$$2(3) + 5y = 16$$

$$6 + 5y = 16$$

$$5y = 10$$

$$y = 2$$

$$\boxed{(3,\ 2)}$$

10. *(Challenge)* $\begin{cases} 4x + 3y = 3 \\ 3x - 5y = 24 \end{cases}$

Step 1: To eliminate y, make the y-coefficients $+15$ and -15: multiply the first equation by 5 and the second equation by 3.

Step 2: Multiply.

$$5(4x + 3y) = 5(3) \quad \Longrightarrow \quad 20x + 15y = 15$$

$$3(3x - 5y) = 3(24) \quad \Longrightarrow \quad 9x - 15y = 72$$

Step 3: Add the two new equations.

$$(20x + 15y) + (9x - 15y) = 15 + 72$$

$$29x = 87$$

$$x = 3$$

Step 4: Substitute $x = 3$ into the first original equation.

$$4(3) + 3y = 3$$

$$12 + 3y = 3$$

$$3y = -9$$

$$y = -3$$

$$\boxed{(3,\ -3)}$$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson · ApexMath
