Solving Linear Systems by Substitution
ApexMath

Notes

What Is a System of Equations?

A **system of equations** is a set of two or more equations that share the same variables. A **solution** to the system is any point (x, y) that makes *every* equation in the system true at the same time.

Geometrically, each equation in a linear system is a straight line. The solution is the point where the two lines **cross** — their intersection.

Three Possible Outcomes

- **One solution:** The two lines cross at exactly one point. This is the most common case.
- **No solution:** The two lines are parallel. They never meet. This happens when the lines have the same slope but different y-intercepts.
- **Infinitely many solutions:** The two equations describe the exact same line. Every point on that line is a solution.

The Substitution Method

The idea is simple: if you can express one variable in terms of the other, you can *plug it in* to the second equation. This turns a two-variable problem into a one-variable problem, which you already know how to solve.

The Four Steps of Substitution

Step 1: Solve one of the equations for one variable. Choose the equation and variable that will be easiest to isolate — look for a variable with a coefficient of 1 or -1.

Step 2: Substitute the expression you found into the *other* equation. Replace every occurrence of that variable with the expression.

Step 3: Solve the resulting equation. It will now have only one variable.

Step 4: Substitute your answer back into the expression from Step 1 to find the value of the second variable. Write your final answer as an ordered pair (x, y).

Always Check Your Answer

Once you have a solution (x, y), substitute both values into *both* original equations and verify that both equations are satisfied. This takes ten seconds and catches errors before they cost you points.

Pro Tip

- Always look for a variable that already has a coefficient of 1 or -1 before you start. Isolating that variable will involve the least amount of fraction arithmetic.
- When you substitute, use parentheses around the entire expression you are plugging in. Forgetting parentheses is the most common source of sign errors.
- After substituting, check that the variable you eliminated is completely gone from the equation. If it is still there, you substituted into the wrong equation.
- If you end up with a statement like $5 = 5$ (always true), the system has infinitely many solutions. If you get $0 = 7$ (never true), there is no solution.
- Write your final answer as an ordered pair (x, y). A bare number with no label is not a complete answer.

Examples

Example 1 (Integer Solution): Solve the system.

$$\begin{cases} y = 2x - 1 \\ 3x + y = 9 \end{cases}$$

Step 1: The first equation is already solved for y.

$$y = 2x - 1$$

Step 2: Substitute $y = 2x - 1$ into the second equation.

Replace y in $3x + y = 9$ with the expression $2x - 1$:

$$3x + (2x - 1) = 9$$

Step 3: Solve for x.

Remove the parentheses and combine like terms:

$$3x + 2x - 1 = 9$$

$$5x - 1 = 9$$
$$5x = 10$$
$$x = 2$$

Step 4: Substitute $x = 2$ back into $y = 2x - 1$.

$$y = 2(2) - 1 = 4 - 1 = 3$$

Check: Equation 1: $y = 2(2) - 1 = 3$ ✓ Equation 2: $3(2) + 3 = 9$ ✓

$$\boxed{(2,\ 3)}$$

Example 2 (Isolate First): Solve the system.

$$\begin{cases} x + 3y = 11 \\ 2x - y = 1 \end{cases}$$

Step 1: Solve the first equation for x.

The variable x has a coefficient of 1, so it is the easiest to isolate:

$$x = 11 - 3y$$

Step 2: Substitute $x = 11 - 3y$ into the second equation.

$$2(11 - 3y) - y = 1$$

Step 3: Expand and solve for y.

$$22 - 6y - y = 1$$

$$22 - 7y = 1$$

$$-7y = -21$$

$$y = 3$$

Step 4: Substitute $y = 3$ back into $x = 11 - 3y$.

$$x = 11 - 3(3) = 11 - 9 = 2$$

Check: Equation 1: $2 + 3(3) = 11$ ✓ Equation 2: $2(2) - 3 = 1$ ✓

$$\boxed{(2,\ 3)}$$

Example 3 (Fractional Solution): Solve the system.

$$\begin{cases} y = 3x + 1 \\ 2x + 4y = 10 \end{cases}$$

Step 1: The first equation is already solved for y.

$$y = 3x + 1$$

Step 2: Substitute into the second equation.

$$2x + 4(3x + 1) = 10$$

Step 3: Expand and solve for x.

$$2x + 12x + 4 = 10$$

$$14x + 4 = 10$$

$$14x = 6$$

$$x = \frac{6}{14} = \frac{3}{7}$$

Step 4: Substitute $x = \dfrac{3}{7}$ back into $y = 3x + 1$.

$$y = 3\left(\frac{3}{7}\right) + 1 = \frac{9}{7} + \frac{7}{7} = \frac{16}{7}$$

Check: Equation 1: $3\left(\frac{3}{7}\right) + 1 = \frac{16}{7}$ ✓ Equation 2: $2\left(\frac{3}{7}\right) + 4\left(\frac{16}{7}\right) = \frac{6}{7} + \frac{64}{7} = \frac{70}{7} = 10$ ✓

$$\boxed{\left(\frac{3}{7}, \frac{16}{7}\right)}$$

Example 4 (No Solution): Solve the system.

$$\begin{cases} y = 2x + 3 \\ 4x - 2y = 6 \end{cases}$$

Step 1: The first equation is already solved for y.

$$y = 2x + 3$$

Step 2: Substitute into the second equation.

$$4x - 2(2x + 3) = 6$$

Step 3: Expand and simplify.

$$4x - 4x - 6 = 6$$

$$-6 = 6$$

This statement is **never true**. No value of x can make -6 equal 6. The two lines are parallel and never intersect.

$$\boxed{\text{No solution}}$$

Example 5 (Infinitely Many Solutions): Solve the system.

$$\begin{cases} y = -x + 4 \\ 2x + 2y = 8 \end{cases}$$

Step 1: The first equation is already solved for y.

$$y = -x + 4$$

Step 2: Substitute into the second equation.

$$2x + 2(-x + 4) = 8$$

Step 3: Expand and simplify.

$$2x - 2x + 8 = 8$$

$$8 = 8$$

This statement is **always true** for any value of x. Both equations describe the same line, so every point on that line is a solution.

$$\boxed{\text{Infinitely many solutions}}$$

Common Mistakes

- **Forgetting parentheses when substituting:** When you substitute an expression for a variable, always wrap the entire expression in parentheses. Writing $2 \cdot 3x + 1$ instead of $2(3x + 1)$ gives a completely different result because the 2 only multiplies the first term.
- **Substituting into the same equation:** After solving for a variable in one equation, always substitute into the *other* equation. Substituting back into the same equation gives a true statement for any value and tells you nothing.
- **Stopping after finding one variable:** Finding x alone is not a complete solution. You must go back and find y as well, then write the answer as an ordered pair (x, y).
- **Not checking the answer:** A quick substitution into both original equations confirms whether your algebra is correct. Skipping this step means a small error can go unnoticed.
- **Misreading special cases:** If the variable disappears and you get a true statement (like $0 = 0$), the answer is infinitely many solutions — not zero solutions. If you get a false statement (like $3 = 7$), the answer is no solution.

Worksheet

Solve each system using the substitution method. Write your answer as an ordered pair (x, y), or state if there is no solution or infinitely many solutions.

1. $\begin{cases} y = x + 2 \\ 2x + y = 8 \end{cases}$

2. $\begin{cases} y = 3x - 5 \\ x + y = 7 \end{cases}$

3. $\begin{cases} x - y = 1 \\ 3x + 2y = 17 \end{cases}$

4. $\begin{cases} 2x + y = 10 \\ x - 3y = -5 \end{cases}$

5. $\begin{cases} y = 4x - 7 \\ 8x - 2y = 14 \end{cases}$

6. $\begin{cases} x + 2y = 6 \\ 3x - y = 5 \end{cases}$

7. $\begin{cases} y = -2x + 5 \\ 4x + 2y = 9 \end{cases}$

8. $\begin{cases} 3x - y = 4 \\ 6x - 2y = 8 \end{cases}$

9. *(Challenge)* $\begin{cases} 2x + 3y = 7 \\ 5x - y = 9 \end{cases}$

10. *(Challenge)* $\begin{cases} \dfrac{x}{2} + y = 4 \\ x - 3y = -6 \end{cases}$

Answer Key

1. $\begin{cases} y = x + 2 \\ 2x + y = 8 \end{cases}$

Step 1: The first equation is already solved for y: $y = x + 2$.

Step 2: Substitute into the second equation.

$$2x + (x + 2) = 8$$

Step 3: Combine like terms and solve for x.

$$3x + 2 = 8$$

$$3x = 6$$

$$x = 2$$

Step 4: Substitute $x = 2$ back into $y = x + 2$.

$$y = 2 + 2 = 4$$

$$\boxed{(2,\ 4)}$$

2. $\begin{cases} y = 3x - 5 \\ x + y = 7 \end{cases}$

Step 1: The first equation is already solved for y: $y = 3x - 5$.

Step 2: Substitute into the second equation.

$$x + (3x - 5) = 7$$

Step 3: Solve for x.

$$4x - 5 = 7$$

$$4x = 12$$

$$x = 3$$

Step 4: Substitute $x = 3$ back into $y = 3x - 5$.

$$y = 3(3) - 5 = 9 - 5 = 4$$

$$\boxed{(3,\ 4)}$$

3. $\begin{cases} x - y = 1 \\ 3x + 2y = 17 \end{cases}$

Step 1: Solve the first equation for x. The coefficient of x is 1.

$$x = y + 1$$

Step 2: Substitute $x = y + 1$ into the second equation.

$$3(y + 1) + 2y = 17$$

Step 3: Expand and solve for y.

$$3y + 3 + 2y = 17$$

$$5y + 3 = 17$$

$$5y = 14$$

$$y = \frac{14}{5}$$

Step 4: Substitute $y = \dfrac{14}{5}$ back into $x = y + 1$.

$$x = \frac{14}{5} + 1 = \frac{14}{5} + \frac{5}{5} = \frac{19}{5}$$

$$\boxed{\left(\frac{19}{5}, \frac{14}{5}\right)}$$

4. $\begin{cases} 2x + y = 10 \\ x - 3y = -5 \end{cases}$

Step 1: Solve the first equation for y.

$$y = 10 - 2x$$

Step 2: Substitute $y = 10 - 2x$ into the second equation.

$$x - 3(10 - 2x) = -5$$

Step 3: Expand and solve for x.

$$\begin{aligned} x - 30 + 6x &= -5 \\ 7x - 30 &= -5 \\ 7x &= 25 \\ x &= \frac{25}{7} \end{aligned}$$

Step 4: Substitute $x = \dfrac{25}{7}$ back into $y = 10 - 2x$.

$$y = 10 - 2\left(\frac{25}{7}\right) = \frac{70}{7} - \frac{50}{7} = \frac{20}{7}$$

$$\boxed{\left(\frac{25}{7}, \frac{20}{7}\right)}$$

5. $\begin{cases} y = 4x - 7 \\ 8x - 2y = 14 \end{cases}$

Step 1: The first equation is already solved for y: $y = 4x - 7$.

Step 2: Substitute into the second equation.

$$8x - 2(4x - 7) = 14$$

Step 3: Expand and simplify.

$$8x - 8x + 14 = 14$$

$$14 = 14$$

This is always true. Both equations represent the same line.

$$\boxed{\text{Infinitely many solutions}}$$

6. $\begin{cases} x + 2y = 6 \\ 3x - y = 5 \end{cases}$

Step 1: Solve the first equation for x.

$$x = 6 - 2y$$

Step 2: Substitute into the second equation.

$$3(6 - 2y) - y = 5$$

Step 3: Expand and solve for y.

$$18 - 6y - y = 5$$

$$18 - 7y = 5$$

$$-7y = -13$$

$$y = \frac{13}{7}$$

Step 4: Substitute $y = \dfrac{13}{7}$ back into $x = 6 - 2y$.

$$x = 6 - 2\left(\frac{13}{7}\right) = \frac{42}{7} - \frac{26}{7} = \frac{16}{7}$$

$$\boxed{\left(\frac{16}{7}, \frac{13}{7}\right)}$$

7. $\begin{cases} y = -2x + 5 \\ 4x + 2y = 9 \end{cases}$

Step 1: The first equation is already solved for y: $y = -2x + 5$.

Step 2: Substitute into the second equation.

$$4x + 2(-2x + 5) = 9$$

Step 3: Expand and simplify.

$$4x - 4x + 10 = 9$$

$$10 = 9$$

This is never true. The two lines are parallel.

$$\boxed{\text{No solution}}$$

8. $\begin{cases} 3x - y = 4 \\ 6x - 2y = 8 \end{cases}$

Step 1: Solve the first equation for y.

$$y = 3x - 4$$

Step 2: Substitute into the second equation.

$$6x - 2(3x - 4) = 8$$

Step 3: Expand and simplify.

$$6x - 6x + 8 = 8$$

$$8 = 8$$

This is always true. Both equations represent the same line.

$$\boxed{\text{Infinitely many solutions}}$$

9. *(Challenge)* $\begin{cases} 2x + 3y = 7 \\ 5x - y = 9 \end{cases}$

Step 1: Solve the second equation for y. The coefficient of y is -1, which makes it easy to isolate.

$$-y = 9 - 5x$$

$$y = 5x - 9$$

Step 2: Substitute $y = 5x - 9$ into the first equation.

$$2x + 3(5x - 9) = 7$$

Step 3: Expand and solve for x.

$$2x + 15x - 27 = 7$$

$$17x - 27 = 7$$

$$17x = 34$$
$$x = 2$$

Step 4: Substitute $x = 2$ back into $y = 5x - 9$.

$$y = 5(2) - 9 = 10 - 9 = 1$$

$$\boxed{(2,\ 1)}$$

10. *(Challenge)* $\begin{cases} \dfrac{x}{2} + y = 4 \\ x - 3y = -6 \end{cases}$

Step 1: Solve the first equation for y. Subtract $\dfrac{x}{2}$ from both sides.

$$y = 4 - \frac{x}{2}$$

Step 2: Substitute into the second equation.

$$x - 3\left(4 - \frac{x}{2}\right) = -6$$

Step 3: Expand the parentheses carefully.

$$x - 12 + \frac{3x}{2} = -6$$

Multiply every term by 2 to clear the fraction.

$$2x - 24 + 3x = -12$$
$$5x - 24 = -12$$
$$5x = 12$$
$$x = \frac{12}{5}$$

Step 4: Substitute $x = \dfrac{12}{5}$ back into $y = 4 - \dfrac{x}{2}$.

$$y = 4 - \frac{1}{2}\left(\frac{12}{5}\right) = 4 - \frac{6}{5} = \frac{20}{5} - \frac{6}{5} = \frac{14}{5}$$

$$\boxed{\left(\frac{12}{5},\ \frac{14}{5}\right)}$$

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