Systems of Inequalities
ApexMath

Notes

What Is a System of Inequalities?

In previous lessons we solved systems of *equations*, where the solution was one or more specific points. A **system of inequalities** is different. Instead of finding exact intersection points, we shade **regions** of the plane. The solution is every point (x, y) that satisfies *all* the inequalities at the same time — the region where all the shaded areas **overlap**.

Graphing a Single Linear Inequality

Before dealing with a system, you must know how to graph one inequality on its own.

Step 1: Replace the inequality sign with $=$ and graph the boundary line.

- Use a **solid line** for $\leq$ or $\geq$ (the boundary is included).
- Use a **dashed line** for $<$ or $>$ (the boundary is not included).

Step 2: Pick a **test point** that is not on the boundary line. The point $(0, 0)$ works perfectly unless the boundary passes through the origin.

Step 3: Substitute the test point into the original inequality.

- If the inequality is **true**, shade the side of the line that contains the test point.
- If the inequality is **false**, shade the *other* side.

Graphing a Single Quadratic Inequality

The process is the same, but the boundary is a parabola instead of a line.

Step 1: Replace the inequality sign with $=$ and graph the boundary parabola. Solid curve for $\leq$ or $\geq$; dashed curve for $<$ or $>$.

Step 2: Pick a test point not on the parabola (the vertex is often convenient).

Step 3: Substitute and shade the correct region — either the interior (inside the cup of the parabola) or the exterior.

Solving a System of Inequalities

Step 1: Graph each inequality on the same coordinate plane, shading its region.

Step 2: Identify the region where *all* shaded areas overlap. This overlapping region is the solution.

Step 3: Verify by picking a test point from inside the overlapping region and confirming it satisfies every inequality in the system.

What Does "No Solution" Look Like?

If the shaded regions for the individual inequalities do not overlap at all, the system has no solution. This is similar to parallel lines in a system of equations — there is no point that satisfies all conditions simultaneously.

Pro Tip

- Always test $(0, 0)$ first — it is the easiest arithmetic. Only choose a different test point if the boundary line or curve passes through the origin.
- Shade lightly or use different patterns for each inequality so you can clearly see where all regions overlap. The solution region is where every inequality's shading is present.
- The solid vs. dashed distinction matters. A solid boundary means points on the line are solutions; a dashed boundary means they are not. Never mix these up.
- After shading, always verify your overlap region with a test point. Plug it into *every* inequality in the system. If any one fails, you have shaded the wrong side.
- For a quadratic inequality like $y \geq x^2$, remember: the region that satisfies this is *inside* the parabola (above the curve, since the parabola opens upward). For $y \leq x^2$ it is the exterior (below the curve). Always confirm with a test point rather than memorising a rule.

Examples

Example 1 (Two Linear Inequalities): Graph the solution to the system.

$$\begin{cases} y > x + 1 \\ y \leq -x + 4 \end{cases}$$

Step 1: Graph the boundary lines.

Inequality 1: $y = x + 1$. Draw a **dashed** line (strict inequality $>$). Slope $= 1$, y-intercept $= 1$.

Inequality 2: $y = -x + 4$. Draw a **solid** line ($\leq$ includes the boundary). Slope $= -1$, y-intercept $= 4$.

The two boundary lines intersect at:

$$x + 1 = -x + 4 \implies 2x = 3 \implies x = \tfrac{3}{2}, \quad y = \tfrac{5}{2}$$

Step 2: Determine which side to shade for each inequality.

Use test point $(0, 0)$:

- $y > x + 1$: $0 > 0 + 1 = 1$. **False.** Shade the side that does *not* contain $(0, 0)$ — above the dashed line.
- $y \leq -x + 4$: $0 \leq -0 + 4 = 4$. **True.** Shade the side that *does* contain $(0, 0)$ — below the solid line.

Step 3: Identify the overlapping region.

The solution is the region that is *above* the dashed line $y = x + 1$ and *below or on* the solid line $y = -x + 4$.

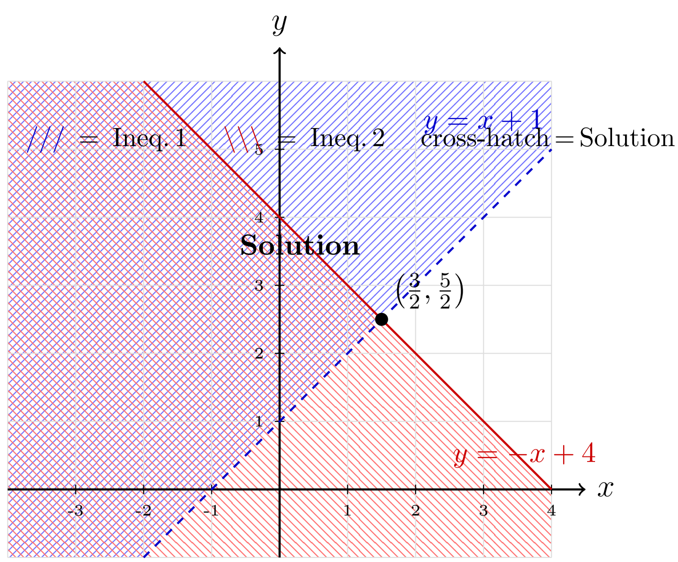
Step 4: Verify with a test point in the overlap.

Try $(0, 3)$:

$$y > x + 1 : \quad 3 > 0 + 1 = 1 \quad \checkmark$$
$$y \leq -x + 4 : \quad 3 \leq -0 + 4 = 4 \quad \checkmark$$

$(0, 3)$ satisfies both. The shading is correct.

Graph of Example 1:



Example 2 (Linear and Quadratic): Graph the solution to the system.

$$\begin{cases} y \geq x^2 \\ y < x + 2 \end{cases}$$

Step 1: Graph the boundaries.

Inequality 1: $y = x^2$. Draw a **solid** parabola ($\geq$ includes the boundary). Vertex at $(0, 0)$, opening upward.

Inequality 2: $y = x + 2$. Draw a **dashed** line ($<$ excludes the boundary). Slope $= 1$, y-intercept $= 2$.

The boundaries intersect where $x^2 = x + 2$:

$$x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0 \implies x = 2 \text{ and } x = -1$$

Intersection points: $(2, 4)$ and $(-1, 1)$.

Step 2: Determine which side to shade.

Use test point $(0, 1)$:

- $y \geq x^2$: $1 \geq 0^2 = 0$. **True.** Shade the side containing $(0, 1)$ — inside (above) the parabola.
- $y < x + 2$: $1 < 0 + 2 = 2$. **True.** Shade the side containing $(0, 1)$ — below the dashed line.

Step 3: Identify the overlapping region.

The solution is the region *inside or on* the parabola $y = x^2$ and *below* the dashed line $y = x + 2$. This is a bounded region between the two curves, from $x = -1$ to $x = 2$.

Step 4: Verify.

$(0, 1)$: $1 \geq 0$ ✓ and $1 < 2$ ✓.

Example 3 (Two Linear, Overlapping Wedge): Graph the solution to the system.

$$\begin{cases} y > 2x - 3 \\ y \geq -x + 1 \end{cases}$$

Step 1: Graph the boundaries.

Inequality 1: $y = 2x - 3$. Draw a **dashed** line. Slope $= 2$, y-intercept $= -3$.

Inequality 2: $y = -x + 1$. Draw a **solid** line. Slope $= -1$, y-intercept $= 1$.

Boundaries intersect at:

$$2x - 3 = -x + 1 \implies 3x = 4 \implies x = \tfrac{4}{3}, \quad y = -\tfrac{1}{3}$$

Step 2: Determine which side to shade.

Use test point $(0, 2)$:

- $y > 2x - 3$: $2 > 2(0) - 3 = -3$. **True.** Shade above the dashed line.
- $y \geq -x + 1$: $2 \geq -0 + 1 = 1$. **True.** Shade above the solid line.

Step 3: Identify the overlapping region.

Both inequalities shade the upper region. The solution is the wedge-shaped region above both lines, pointing upward from their intersection at $\left(\frac{4}{3}, -\frac{1}{3}\right)$.

Step 4: Verify.

$(0, 2)$: $2 > -3$ ✓ and $2 \geq 1$ ✓.

Example 4 (Quadratic and Linear): Graph the solution to the system.

$$\begin{cases} y \leq -x^2 + 4 \\ y > x - 2 \end{cases}$$

Step 1: Graph the boundaries.

Inequality 1: $y = -x^2 + 4$. Draw a **solid** downward parabola. Vertex at $(0, 4)$.

Inequality 2: $y = x - 2$. Draw a **dashed** line. Slope $= 1$, y-intercept $= -2$.

Boundaries intersect where $-x^2 + 4 = x - 2$:

$$x^2 + x - 6 = 0 \implies (x + 3)(x - 2) = 0 \implies x = -3 \text{ and } x = 2$$

Intersection points: $(-3, -5)$ and $(2, 0)$.

Step 2: Determine which side to shade.

Use test point $(0, 1)$:

- $y \leq -x^2 + 4$: $1 \leq -0 + 4 = 4$. **True.** Shade inside (below or on) the downward parabola.
- $y > x - 2$: $1 > 0 - 2 = -2$. **True.** Shade above the dashed line.

Step 3: Identify the overlapping region.

The solution is the region *below or on* the solid parabola and *above* the dashed line, bounded between $x = -3$ and $x = 2$.

Step 4: Verify.

$(0, 1)$: $1 \leq 4$ ✓ and $1 > -2$ ✓.

Example 5 (Two Quadratic Inequalities): Graph the solution to the system.

$$\begin{cases} y < x^2 + 1 \\ y > -x^2 + 3 \end{cases}$$

Step 1: Graph the boundaries.

Inequality 1: $y = x^2 + 1$. Draw a **dashed** upward parabola. Vertex at $(0, 1)$.

Inequality 2: $y = -x^2 + 3$. Draw a **dashed** downward parabola. Vertex at $(0, 3)$.

Boundaries intersect where $x^2 + 1 = -x^2 + 3$:

$$2x^2 = 2 \implies x^2 = 1 \implies x = \pm 1$$

Intersection points: $(1, 2)$ and $(-1, 2)$.

Step 2: Determine which side to shade.

Use test point $(2, 1)$:

- $y < x^2 + 1$: $1 < 4 + 1 = 5$. **True.** Shade below (outside) the dashed upward parabola.
- $y > -x^2 + 3$: $1 > -4 + 3 = -1$. **True.** Shade above (outside) the dashed downward parabola.

Step 3: Identify the overlapping region.

The solution is the two regions to the *left* of $x = -1$ and to the *right* of $x = 1$ — the exterior region where both parabolas' outsides overlap. The area between the two parabolas (from $x = -1$ to $x = 1$) is *not* part of the solution.

Step 4: Verify.

$(2, 1)$: $1 < 5$ ✓ and $1 > -1$ ✓.

Common Mistakes

- **Using solid vs. dashed incorrectly:** A strict inequality ($<$ or $>$) means the boundary is *not* part of the solution — draw a dashed line or curve. A non-strict inequality ($\leq$ or $\geq$) means the boundary *is* included — draw a solid line or curve. Mixing these up changes the answer.
- **Shading the wrong side:** Always use a test point. Do not try to guess which side to shade by looking at the inequality symbol alone. Substitute the test point and let the arithmetic decide.
- **Forgetting to verify the overlap region:** After shading both inequalities, always pick a point from the overlapping region and check it against every inequality. One incorrect shade can go unnoticed without this step.
- **Choosing a test point on the boundary:** The test point must be strictly off the boundary line or curve. A point on the boundary gives $0 = 0$ or similar, which is inconclusive.
- **Not recognising a quadratic inequality's shaded region:** For $y \geq x^2$ (upward parabola), the solution is the region inside the cup — above the curve. For $y \leq -x^2 + k$ (downward parabola), the solution is below the curve. Always verify with a test point rather than assuming.

Worksheet

For each system, describe (or sketch) the solution region. Identify the boundary type (solid or dashed), the side to shade for each inequality, and verify with a test point.

1. $\begin{cases} y > x + 1 \\ y < -x + 5 \end{cases}$

2. $\begin{cases} y \leq 2x - 1 \\ y > -x + 2 \end{cases}$

3. $\begin{cases} y \geq x^2 - 1 \\ y \leq 3 \end{cases}$

4. $\begin{cases} y < -x^2 + 5 \\ y > 1 \end{cases}$

5. $\begin{cases} y \geq -x + 4 \\ y \leq 2x - 2 \end{cases}$

6. $\begin{cases} y > x^2 - 3 \\ y < -x^2 + 5 \end{cases}$

7. $\begin{cases} y < x + 3 \\ y \geq x - 1 \end{cases}$

8. *(Challenge)* $\begin{cases} y \le x^2 + 2x \\ y > x + 2 \end{cases}$

Answer Key

For each problem the answer describes the boundary lines or curves, which side to shade for each inequality, the overlap region, and a verified test point.

1. $\begin{cases} y > x + 1 \\ y < -x + 5 \end{cases}$

Boundaries: $y = x + 1$ (dashed, strict $>$) and $y = -x + 5$ (dashed, strict $<$). They intersect at:

$$x + 1 = -x + 5 \implies 2x = 4 \implies x = 2, \quad y = 3$$

Shading: Test point $(0, 0)$.

$$y > x + 1: \quad 0 > 1 \quad \text{False} \implies \text{shade above } y = x + 1$$

$$y < -x + 5: \quad 0 < 5 \quad \text{True} \implies \text{shade below } y = -x + 5$$

Solution: The region above the dashed line $y = x + 1$ and below the dashed line $y = -x + 5$ — a triangular wedge opening upward, with its tip at $(2, 3)$.

Verify with $(1, 3)$:

$$3 > 1 + 1 = 2 \quad \checkmark \qquad 3 < -1 + 5 = 4 \quad \checkmark$$

$$\boxed{\text{Region above } y = x + 1 \text{ and below } y = -x + 5, \text{ tip at } (2,\,3)}$$

2. $\begin{cases} y \le 2x - 1 \\ y > -x + 2 \end{cases}$

Boundaries: $y = 2x - 1$ (solid, $\le$) and $y = -x + 2$ (dashed, strict $>$). They intersect at:

$$2x - 1 = -x + 2 \implies 3x = 3 \implies x = 1, \quad y = 1$$

Shading: Test point $(0, 0)$.

$$y \le 2x - 1: \quad 0 \le -1 \quad \text{False} \implies \text{shade below } y = 2x - 1$$

$$y > -x + 2: \quad 0 > 2 \quad \text{False} \implies \text{shade above } y = -x + 2$$

Solution: The region below or on the solid line $y = 2x - 1$ and above the dashed line $y = -x + 2$, to the right of their intersection $(1, 1)$.

Verify with $(2, 2)$:

$$2 \le 2(2) - 1 = 3 \quad \checkmark \qquad 2 > -2 + 2 = 0 \quad \checkmark$$

$$\boxed{\text{Region below/on } y = 2x - 1 \text{ and above } y = -x + 2,\ \text{vertex at } (1,\,1)}$$

3. $\begin{cases} y \geq x^2 - 1 \\ y \leq 3 \end{cases}$

Boundaries: $y = x^2 - 1$ (solid upward parabola, $\geq$) and $y = 3$ (solid horizontal line, $\leq$). They intersect where:

$$x^2 - 1 = 3 \implies x^2 = 4 \implies x = 2 \text{ and } x = -2$$

Intersection points: $(2,3)$ and $(-2,3)$.

Shading: Test point $(0,1)$.

$$y \geq x^2 - 1: \quad 1 \geq -1 \quad \text{True} \implies \text{shade above/inside the parabola}$$

$$y \leq 3: \quad 1 \leq 3 \quad \text{True} \implies \text{shade below the line}$$

Solution: The bounded region inside the parabola (above it) and below or on the line $y = 3$, between $x = -2$ and $x = 2$.

Verify with $(0,1)$:

$$1 \geq 0 - 1 = -1 \quad \checkmark \qquad 1 \leq 3 \quad \checkmark$$

$$\boxed{\text{Bounded region inside parabola and below } y = 3,\ \text{between } (-2,3) \text{ and } (2,3)}$$

4. $\begin{cases} y < -x^2 + 5 \\ y > 1 \end{cases}$

Boundaries: $y = -x^2 + 5$ (dashed downward parabola, vertex at $(0,5)$) and $y = 1$ (dashed horizontal line). They intersect where:

$$-x^2 + 5 = 1 \implies x^2 = 4 \implies x = 2 \text{ and } x = -2$$

Intersection points: $(2,1)$ and $(-2,1)$.

Shading: Test point $(0,3)$.

$$y < -x^2 + 5: \quad 3 < 5 \quad \text{True} \implies \text{shade below (inside) the dashed parabola}$$

$$y > 1: \quad 3 > 1 \quad \text{True} \implies \text{shade above the dashed line}$$

Solution: The bounded region below the dashed downward parabola and above the dashed line $y = 1$, between $x = -2$ and $x = 2$. No boundary points are included.

Verify with $(0,3)$:

$$3 < -0 + 5 = 5 \quad \checkmark \qquad 3 > 1 \quad \checkmark$$

$$\boxed{\text{Bounded region inside parabola and above } y = 1, \text{ strictly between } (-2,1) \text{ and } (2,1)}$$

5. $\begin{cases} y \geq -x + 4 \\ y \leq 2x - 2 \end{cases}$

Boundaries: $y = -x + 4$ (solid, $\geq$) and $y = 2x - 2$ (solid, $\leq$). They intersect at:

$$-x + 4 = 2x - 2 \implies -3x = -6 \implies x = 2, \quad y = 2$$

Shading: Test point $(0, 0)$.

$$y \geq -x + 4 : \quad 0 \geq 4 \quad \text{False} \implies \text{shade above/right of } y = -x + 4$$

$$y \leq 2x - 2 : \quad 0 \leq -2 \quad \text{False} \implies \text{shade below/right of } y = 2x - 2$$

Solution: The region to the right of their intersection $(2, 2)$, above or on $y = -x + 4$ and below or on $y = 2x - 2$ — a wedge opening to the right.

Verify with $(3, 2)$:

$$2 \geq -3 + 4 = 1 \quad \checkmark \qquad 2 \leq 2(3) - 2 = 4 \quad \checkmark$$

$$\boxed{\text{Wedge region to the right of (2, 2), between the two solid lines}}$$

6. $\begin{cases} y > x^2 - 3 \\ y < -x^2 + 5 \end{cases}$

Boundaries: $y = x^2 - 3$ (dashed upward parabola, vertex $(0, -3)$) and $y = -x^2 + 5$ (dashed downward parabola, vertex $(0, 5)$). They intersect where:

$$x^2 - 3 = -x^2 + 5 \implies 2x^2 = 8 \implies x = 2 \text{ and } x = -2$$

Intersection points: $(2, 1)$ and $(-2, 1)$.

Shading: Test point $(0, 0)$.

$$y > x^2 - 3 : \quad 0 > -3 \quad \text{True} \implies \text{shade above/inside the upward parabola}$$

$$y < -x^2 + 5 : \quad 0 < 5 \quad \text{True} \implies \text{shade below/inside the downward parabola}$$

Solution: The lens-shaped region between the two dashed parabolas, from $x = -2$ to $x = 2$, strictly between the curves.

Verify with $(0, 0)$:

$$0 > -3 \quad \checkmark \qquad 0 < 5 \quad \checkmark$$

$$\boxed{\text{Lens-shaped region between the two parabolas, strictly between } (-2,1) \text{ and } (2,1)}$$

7. $\begin{cases} y < x+3 \\ y \geq x-1 \end{cases}$

Boundaries: $y = x+3$ (dashed, strict $<$) and $y = x-1$ (solid, $\geq$). Both lines have slope 1 — they are **parallel**.

Parallel lines never intersect, so the two shaded regions form an infinite horizontal strip between them.

Shading: Test point $(0,1)$.

$$y < x+3: \quad 1 < 3 \quad \text{True} \implies \text{shade below the dashed line}$$

$$y \geq x-1: \quad 1 \geq -1 \quad \text{True} \implies \text{shade above/on the solid line}$$

Solution: The infinite strip between the two parallel lines — below the dashed line $y = x+3$ and above or on the solid line $y = x-1$ — for all values of x.

Verify with $(0,1)$:

$$1 < 3 \quad \checkmark \qquad 1 \geq -1 \quad \checkmark$$

$$\boxed{\text{Infinite strip between parallel lines: on/above } y = x-1 \text{ and below } y = x+3}$$

8. *(Challenge)* $\begin{cases} y \leq x^2+2x \\ y > x+2 \end{cases}$

Boundaries: $y = x^2+2x$ (solid upward parabola, vertex at $(-1,-1)$) and $y = x+2$ (dashed line, slope 1, y-intercept 2). They intersect where:

$$x^2+2x = x+2 \implies x^2+x-2 = 0 \implies (x+2)(x-1) = 0$$

$$x = -2, \quad y = 0 \qquad \text{and} \qquad x = 1, \quad y = 3$$

Intersection points: $(-2,0)$ and $(1,3)$.

Shading: Test point $(0,0)$.

$$y \leq x^2+2x: \quad 0 \leq 0 \quad \text{True} \implies \text{shade below/on the solid parabola}$$

$$y > x+2: \quad 0 > 2 \quad \text{False} \implies \text{shade above the dashed line}$$

$(0,0)$ only satisfies the first inequality, so it is *not* in the overlap region. We need a point that is below the parabola *and* above the line.

Test $(3, 10)$:

$$y \leq x^2 + 2x : \quad 10 \leq 9 + 6 = 15 \quad \checkmark$$
$$y > x + 2 : \quad 10 > 5 \quad \checkmark$$

Solution: The two separate regions where the parabola dips below the line — to the left of $(-2, 0)$ and to the right of $(1, 3)$. The region between the two intersection points does not satisfy both inequalities simultaneously.

$$\boxed{\text{Two exterior regions: left of } (-2, 0) \text{ and right of } (1, 3)\text{, below parabola and above line}}$$

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End of Lesson · ApexMath
