Applications of Sine and Cosine Laws
ApexMath

Notes

From Formulas to the Real World

Every formula in this unit was built to solve real problems. In this lesson, the same laws you have been practising are applied to navigation, surveying, indirect measurement, and geometry. The mathematics does not change — the challenge is reading a word problem, drawing a labelled diagram, identifying the configuration, and choosing the right law.

Setting Up Word Problems: A Process

- **Step 1 — Draw a diagram.** Sketch the situation. Label every known side and angle. Mark the unknown clearly.
- **Step 2 — Identify the triangle.** Which three points form the triangle? What sides and angles are known?
- **Step 3 — Classify the configuration.** Is it AAS, ASA, SSA, SAS, or SSS? Which law applies?
- **Step 4 — Solve.** Apply the law, show every step, and include units in the final answer.
- **Step 5 — Check.** Does the answer make sense in context? A distance cannot be negative. An angle cannot exceed 180. Verify with the angle sum.

Bearings

A **bearing** describes a direction as an angle measured clockwise from north. Bearings are written in the form Nx°E or Nx°W.

- **N40°E** means start facing north, rotate 40 toward the east.
- **N70°W** means start facing north, rotate 70 toward the west.

When two legs of a journey are given as bearings, the **angle between the two paths** is found by subtracting the two bearing angles. This angle becomes the included angle in a SAS triangle.

Area of a Triangle Using Sine

When two sides and their included angle are known, the area can be found without needing the height explicitly:

$$\text{Area} = \frac{1}{2}\, b\, c\, \sin A$$

where b and c are two known sides and A is the angle between them. This formula works for any triangle — not just right triangles.

Pro Tip

- Draw the diagram before reading the problem a second time. Sketching forces you to organize the information spatially, which almost always makes the configuration obvious.
- In bearing problems, the angle between two travel legs is simply the difference in their bearing angles. For example, a leg on N35°E followed by a leg on N80°E creates an included angle of $80 - 35 = 45$ between the two paths. Label this at the turning point — it is the angle in your SAS triangle.
- For indirect measurement problems (finding a height or distance you cannot measure directly), the triangle is formed by the two observation points and the target. Label the observation baseline as one side and the two measured angles as the angles at each end.
- Always include units in every step, not just the final answer. Writing "$a = 14.75$" with no unit is an incomplete answer. Write "$a \approx 14.75$ km" throughout.
- When the area formula gives a surprisingly large or small number, recheck that the angle used is the one *between* the two sides. Using a non-included angle in $\frac{1}{2}bc\sin A$ gives the wrong area.

Examples

Example 1 (Indirect Height): An observer stands 80 m from the base of a cliff. The angle of elevation to the top of the cliff is 34. How tall is the cliff?

Step 1: Draw the triangle. The base is 80 m, the right angle is at the base of the cliff, and the elevation angle is 34 at the observer.

Step 2: This is a right triangle. Use SOHCAHTOA.

$$\tan 34 = \frac{\text{height}}{80}$$

Step 3: Solve for the height.

$$\text{height} = 80 \times \tan 34 = 80 \times 0.6745$$

$$\boxed{\text{height} \approx 53.96 \text{ m}}$$

Example 2 (Bearing — Distance Between Two Points): A ship leaves port and travels 40 km on a bearing of N30°E. It then changes course and travels 55 km on a bearing of N70°E. How far is the ship from port?

Step 1: The angle between the two legs is $70 - 30 = 40$. This is the included angle at the turning point, giving a SAS triangle with sides 40 km and 55 km.

Step 2: Apply the Cosine Law.

$$d^2 = 40^2 + 55^2 - 2(40)(55)\cos 40$$

$$d^2 = 1600 + 3025 - 4400 \times 0.7660$$

$$d^2 = 4625 - 3370.40 = 1254.60$$

Step 3: Take the square root.

$$d = \sqrt{1254.60}$$

$$\boxed{d \approx 35.42 \text{ km from port}}$$

Example 3 (Two Observers — Indirect Distance): Two fire lookouts A and B are 15 km apart. Both spot a forest fire at point F. The angle at A between the line AB and the fire is 52. The angle at B is 63. How far is each lookout from the fire?

Step 1: Find the angle at the fire.

$$F = 180 - 52 - 63 = 65$$

Step 2: This is AAS. Apply the Sine Law. Side $AB = 15$ km is opposite angle F.

$$AF = \frac{15 \times \sin 63}{\sin 65} = \frac{15 \times 0.8910}{0.9063}$$

$$AF \approx 14.75 \text{ km}$$

$$BF = \frac{15 \times \sin 52}{\sin 65} = \frac{15 \times 0.7880}{0.9063}$$

$$\boxed{AF \approx 14.75 \text{ km}, \quad BF \approx 13.04 \text{ km}}$$

Example 4 (Area of a Triangle): A triangular piece of land has two sides of 14 m and 20 m with an included angle of 65. Find the area of the land.

Step 1: Identify the known values. Two sides and the included angle — use the area formula directly.

$$\text{Area} = \frac{1}{2}\, b\, c\, \sin A = \frac{1}{2} \times 14 \times 20 \times \sin 65$$

Step 2: Evaluate.

$$\text{Area} = 140 \times 0.9063$$

$$\boxed{\text{Area} \approx 126.88 \text{ m}^2}$$

Example 5 (Multi-Step — Surveying): A surveyor needs to find the distance across a lake from point A to point C. She sets up a baseline $AB = 500$ m on one shore. From A, the angle to C is 72. From B, the angle to C is 61. Find the distance AC.

Step 1: Find the angle at C.

$$C = 180 - 72 - 61 = 47$$

Step 2: AAS — use the Sine Law. Side $AB = 500$ m is opposite angle C.

$$AC = \frac{500 \times \sin 61}{\sin 47} = \frac{500 \times 0.8746}{0.7314}$$

$$\boxed{AC \approx 597.95 \text{ m}}$$

Common Mistakes

- **Not drawing a diagram.** In application problems, a missing or incorrect diagram almost always leads to a wrong answer. Even a rough sketch with labels prevents most setup errors.
- **Using the wrong angle in bearing problems.** The angle in the triangle is the angle *between the two travel legs*, not the bearing itself. Subtract the two bearing angles to find the included angle.
- **Forgetting units.** Every answer in an application problem must include units. A distance without "km" or "m" is an incomplete answer.
- **Using $\frac{1}{2}bh$ when the height is not given.** If you know two sides and an included angle but not the perpendicular height, use $\frac{1}{2}bc \sin A$. Trying to find the height first is an unnecessary extra step.
- **Misidentifying which angle is at which vertex.** In two-observer problems, the angle at each observer is between the baseline and the line of sight to the target. Label each vertex explicitly on the diagram before writing any formula.

Worksheet

1. Two ships leave the same port. Ship A travels 25 km on a bearing of N35°E. Ship B travels 40 km on a bearing of N80°E. How far apart are the two ships?

2. A triangular hiking trail has legs $AB = 8$ km and $BC = 6$ km. The angle at B between the two legs is 110. Find the straight-line distance AC.

3. A surveyor wants to find the distance across a canyon from point A to point C. She measures a baseline $AB = 500$ m. From A, the angle to C is 72. From B, the angle to C is 61. Find distances AC and BC.

4. A triangular garden has two sides of 18 m and 24 m meeting at an angle of 55. Find the area of the garden.

5. A hiker walks 6 km on a bearing of N25°E, then turns and walks 9 km on a bearing of N70°E. How far is the hiker from the starting point?

6. Three towns form a triangle. The distances between them are 80 km, 65 km, and 90 km. Find the largest angle of the triangle.

7. Two observers stand 200 m apart on flat ground, both watching a hot air balloon directly above the line between them. The angle of elevation from observer A is 71 and from observer B is 53. Find the height of the balloon above the ground.
(Hint: First find the slant distance from A to the balloon using the Sine Law, then find the vertical height.)

8. **(Challenge)** A parallelogram has sides of 12 cm and 18 cm. The angle between the sides is 65. Find the lengths of both diagonals.
(Hint: A parallelogram has two triangles. The included angle for the second diagonal is $180 - 65 = 115$.)

Answer Key

1. Ship A: 25 km on N35°E. Ship B: 40 km on N80°E.

Step 1: Find the angle between the two paths.

$$\theta = 80 - 35 = 45$$

Step 2: SAS triangle with sides 25 km and 40 km and included angle 45**. Apply the Cosine Law.**

$$d^2 = 25^2 + 40^2 - 2(25)(40)\cos 45 = 625 + 1600 - 2000 \times 0.7071 = 2225 - 1414.21 = 810.79$$

Step 3: Take the square root.

$$d = \sqrt{810.79}$$

$$\boxed{d \approx 28.47 \text{ km}}$$

2. $AB = 8$ km, $BC = 6$ km, angle at $B = 110$.

Step 1: SAS. Apply the Cosine Law. Note $\cos 110 \approx -0.3420$**.**

$$AC^2 = 8^2 + 6^2 - 2(8)(6)\cos 110 = 64 + 36 - 96 \times (-0.3420) = 100 + 32.83 = 132.83$$

Step 2: Take the square root.

$$AC = \sqrt{132.83}$$

$$\boxed{AC \approx 11.53 \text{ km}}$$

3. $AB = 500$ m, angle at $A = 72$, angle at $B = 61$.

Step 1: Find the angle at C**.**

$$C = 180 - 72 - 61 = 47$$

Step 2: AAS. Side $AB = 500$ **m is opposite** C**. Find** AC**.**

$$AC = \frac{500 \times \sin 61}{\sin 47} = \frac{500 \times 0.8746}{0.7314} \approx 597.95 \text{ m}$$

Step 3: Find BC**.**

$$BC = \frac{500 \times \sin 72}{\sin 47} = \frac{500 \times 0.9511}{0.7314} \approx 650.20 \text{ m}$$

$$\boxed{AC \approx 597.95 \text{ m}, \quad BC \approx 650.20 \text{ m}}$$

4. Two sides 18 m and 24 m, included angle 55.

Step 1: Apply the area formula.

$$\text{Area} = \frac{1}{2} \times 18 \times 24 \times \sin 55 = 216 \times 0.8192$$

$$\boxed{\text{Area} \approx 176.94 \text{ m}^2}$$

5. 6 km on N25°E, then 9 km on N70°E.

Step 1: Angle between paths.

$$\theta = 70 - 25 = 45$$

Step 2: SAS. Apply the Cosine Law.

$$d^2 = 6^2 + 9^2 - 2(6)(9)\cos 45 = 36 + 81 - 108 \times 0.7071 = 117 - 76.37 = 40.63$$

Step 3: Take the square root.

$$d = \sqrt{40.63}$$

$$\boxed{d \approx 6.37 \text{ km from the starting point}}$$

6. Sides 80 km, 65 km, 90 km. Find the largest angle.

Step 1: Largest side is 90 km. Let $c = 90$, $a = 65$, $b = 80$. Find C.

$$\cos C = \frac{65^2 + 80^2 - 90^2}{2(65)(80)} = \frac{4225 + 6400 - 8100}{10400} = \frac{2525}{10400} = 0.2428$$

$$C = \cos^{-1}(0.2428)$$

$$\boxed{C \approx 75.95}$$

7. Two observers 200 m apart. Elevation angles 71 and 53.

Step 1: Form a triangle with the two observers (A, B) and the balloon (F). The angles at A and B are elevation angles. Find the angle at F.

$$F = 180 - 71 - 53 = 56$$

Step 2: AAS. Side $AB = 200$ m is opposite angle F. Find the slant distance AF.

$$AF = \frac{200 \times \sin 53}{\sin 56} = \frac{200 \times 0.7986}{0.8290} \approx 192.67 \text{ m}$$

Step 3: Find the height of the balloon. The vertical height is the component of AF in the upward direction.

$$h = AF \times \sin 71 = 192.67 \times 0.9455$$

$$\boxed{h \approx 182.17 \text{ m}}$$

8. (Challenge) Parallelogram: sides 12 cm and 18 cm, angle 65.

Step 1: Find diagonal 1. The included angle in this triangle is 65.

$$d_1^2 = 12^2 + 18^2 - 2(12)(18)\cos 65 = 144 + 324 - 432 \times 0.4226 = 468 - 182.56 = 285.44$$

$$d_1 = \sqrt{285.44} \approx 16.89 \text{ cm}$$

Step 2: Find diagonal 2. In a parallelogram, consecutive angles are supplementary, so the included angle for the second triangle is $180 - 65 = 115$.

$$d_2^2 = 12^2 + 18^2 - 2(12)(18)\cos 115 = 144 + 324 - 432 \times (-0.4226) = 468 + 182.56 = 650.56$$

$$d_2 = \sqrt{650.56} \approx 25.51 \text{ cm}$$

$$\boxed{d_1 \approx 16.89 \text{ cm}, \quad d_2 \approx 25.51 \text{ cm}}$$

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