Choosing the Right Law
ApexMath

Notes

What This Lesson Is About

You now have three tools for solving triangles: the Sine Law, the Ambiguous Case (a special application of the Sine Law), and the Cosine Law. The challenge is no longer how to use each tool — it is knowing *which* tool to reach for when a problem does not tell you. This lesson builds that decision-making skill.

The Four Configurations — A Full Review

Every triangle problem gives you a combination of known sides and angles. The combination tells you exactly which law to use.

- **AAS — Angle, Angle, Side (non-included):** Two angles and a side that is *not* between them. You always have a matching side-angle pair. **Use the Sine Law.** Find the third angle first with $A + B + C = 180$, then apply the law.
- **ASA — Angle, Side, Angle (included):** Two angles and the side *between* them. You always have a matching side-angle pair. **Use the Sine Law.** Same process as AAS.
- **SSA — Side, Side, Angle (non-included):** Two sides and an angle opposite one of them. **Use the Sine Law — but check for the ambiguous case first** by computing $h = b \sin A$. There may be zero, one, or two triangles.
- **SAS — Side, Angle, Side (included):** Two sides and the angle *between* them. No matching pair exists, so the Sine Law cannot start the problem. **Use the Cosine Law** to find the missing side, then use the Sine Law for the remaining angles.
- **SSS — Side, Side, Side:** All three sides known, no angles. No matching pair exists. **Use the Cosine Law** to find the largest angle first, then use the Sine Law or angle sum for the rest.

Decision Table

Given	Law to Use	Watch Out For
AAS	Sine Law	Find third angle first
ASA	Sine Law	Find third angle first
SSA	Sine Law	Check ambiguous case ($h = b \sin A$)
SAS	Cosine Law $\rightarrow$ Sine Law	Negative cosine if $A > 90$
SSS	Cosine Law $\rightarrow$ Sine Law	Find largest angle first

The Two-Step Pattern

Many problems require two laws in sequence:

1. Use the **Cosine Law** to find the missing side (SAS) or the largest angle (SSS).
2. Use the **Sine Law** to find the remaining unknowns, since you now have a matching side-angle pair.

This Cosine-then-Sine pattern is the standard approach for SAS and SSS problems. Recognizing it early saves time.

Pro Tip

- Before writing a single formula, count your knowns and ask: *do I have a matching side-angle pair?* If yes, the Sine Law can start the problem. If no, you need the Cosine Law.
- The word "between" is the key to distinguishing SAS from SSA and ASA from AAS. The included side or angle is always sandwiched between the other two known values.
- When a problem gives you three sides and asks for all angles, always find the angle opposite the *longest* side first using the Cosine Law. This eliminates any ambiguity about whether the triangle has an obtuse angle before you use the Sine Law.
- After finding your first unknown, stop and reassess. You may now have a matching pair that lets you switch to the Sine Law, which is faster than applying the Cosine Law a second time.
- Every completed triangle should satisfy two checks: $A + B + C = 180$ and the Sine Law ratios $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ should all be equal. If either check fails, find the error before moving on.

Examples

Example 1 (AAS — Sine Law): In triangle ABC, $A = 52$, $B = 73$, and $a = 18$. Find sides b and c.

Step 1: Identify the configuration. Two angles and a non-included side — this is **AAS**. Use the Sine Law.

Step 2: Find the third angle.

$$C = 180 - 52 - 73 = 55$$

Step 3: Find b using the known pair (a, A).

$$b = \frac{a \sin B}{\sin A} = \frac{18 \times \sin 73}{\sin 52} = \frac{18 \times 0.9563}{0.7880}$$

$$b \approx 21.84$$

Step 4: Find c.

$$c = \frac{18 \times \sin 55}{\sin 52} = \frac{18 \times 0.8192}{0.7880}$$

$$\boxed{b \approx 21.84, \quad c \approx 18.71}$$

Example 2 (SAS — Cosine Law, then Sine Law): In triangle ABC, $b = 10$, $c = 14$, and $A = 65$. Find all missing sides and angles.

Step 1: Identify the configuration. Two sides with the angle *between* them — this is **SAS**. Use the Cosine Law first.
Step 2: Find side a.

$$a^2 = 10^2 + 14^2 - 2(10)(14)\cos 65 = 100 + 196 - 280 \times 0.4226 = 296 - 118.33 = 177.67$$

$$a = \sqrt{177.67} \approx 13.33$$

Step 3: Now use the Sine Law to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{10 \times \sin 65}{13.33} = \frac{10 \times 0.9063}{13.33} = 0.6798$$

$$B = \sin^{-1}(0.6798) \approx 42.84$$

Step 4: Find C.

$$C = 180 - 65 - 42.84 = 72.16$$

$$\boxed{a \approx 13.33, \quad B \approx 42.84, \quad C \approx 72.16}$$

Example 3 (SSS — Cosine Law, then Sine Law): In triangle ABC, $a = 8$, $b = 11$, $c = 14$. Find all three angles.

Step 1: Identify the configuration. All three sides known — this is **SSS**. Use the Cosine Law to find the largest angle first (C, opposite $c = 14$).
Step 2: Find C.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 121 - 196}{2(8)(11)} = \frac{-11}{176} = -0.0625$$

$$C = \cos^{-1}(-0.0625) \approx 93.58$$

Step 3: Use the Sine Law to find B.

$$\sin B = \frac{b \sin C}{c} = \frac{11 \times \sin 93.58}{14} = \frac{11 \times 0.9980}{14} = 0.7843$$

$$B = \sin^{-1}(0.7843) \approx 51.64$$

Step 4: Find A.

$$A = 180 - 93.58 - 51.64 = 34.78$$

$$\boxed{A \approx 34.78, \quad B \approx 51.64, \quad C \approx 93.58}$$

Example 4 (SSA — Ambiguous Case): In triangle ABC, $a = 10$, $b = 15$, and $A = 38$. Solve all possible triangles.

Step 1: Identify the configuration. Two sides and a non-included angle — this is **SSA**. Check for the ambiguous case first.

$$h = b \sin A = 15 \times \sin 38 = 15 \times 0.6157 = 9.235$$

Since $h = 9.235 \leq a = 10 < b = 15$, **two triangles** exist.
Step 2: Find $\sin B$.

$$\sin B = \frac{b \sin A}{a} = \frac{15 \times \sin 38}{10} = 0.9236$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9236) \approx 67.44, \quad C_1 = 180 - 38 - 67.44 = 74.56$$

$$c_1 = \frac{10 \times \sin 74.56}{\sin 38} = \frac{10 \times 0.9639}{0.6157} \approx 15.66$$

Triangle 2:

$$B_2 = 180 - 67.44 = 112.56, \quad C_2 = 180 - 38 - 112.56 = 29.44$$

$$c_2 = \frac{10 \times \sin 29.44}{\sin 38} = \frac{10 \times 0.4915}{0.6157} \approx 7.98$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 67.44,\ C_1 \approx 74.56,\ c_1 \approx 15.66 \\ \textbf{Triangle 2:} & B_2 \approx 112.56,\ C_2 \approx 29.44,\ c_2 \approx 7.98 \end{array}}$$

Example 5 (SAS with Obtuse Angle — Two-Step): In triangle ABC, $b = 12$, $c = 9$, and $A = 110$. Find all missing sides and angles.

Step 1: SAS with an obtuse included angle. $\cos 110 \approx -0.3420$**.**

$$a^2 = 12^2 + 9^2 - 2(12)(9)\cos 110 = 144 + 81 - 216 \times (-0.3420) = 225 + 73.87 = 298.87$$

$$a = \sqrt{298.87} \approx 17.29$$

Step 2: Sine Law to find B (use smaller side to avoid ambiguity with the obtuse angle already found).

$$\sin B = \frac{b \sin A}{a} = \frac{12 \times \sin 110}{17.29} = \frac{12 \times 0.9397}{17.29} = 0.6521$$

$$B = \sin^{-1}(0.6521) \approx 40.71$$

Step 3: Find C.

$$C = 180 - 110 - 40.71 = 29.29$$

$$\boxed{a \approx 17.29, \quad B \approx 40.71, \quad C \approx 29.29}$$

Common Mistakes

- **Using the Sine Law for SAS.** Without a matching side-angle pair, the Sine Law has nothing to work with. Always check for a pair first. SAS and SSS must begin with the Cosine Law.
- **Forgetting to check for the ambiguous case in SSA.** Every SSA problem requires computing $h = b \sin A$ before applying the Sine Law. Jumping straight to the formula risks finding only one triangle when two exist, or reporting a triangle that does not exist at all.
- **Misidentifying SAS vs. SSA.** The difference is whether the known angle is *between* the two known sides (SAS) or *opposite* one of them (SSA). Draw and label the triangle before deciding.
- **Applying the Cosine Law twice when once is enough.** After using the Cosine Law to find the first unknown in a SAS or SSS problem, switch to the Sine Law. It is simpler and faster for the remaining unknowns.
- **Not verifying the answer.** After completing any triangle, check that all angles sum to 180. For extra confidence, verify that the Sine Law ratios are consistent.

Worksheet

For each problem, first identify the configuration (AAS, ASA, SSA, SAS, or SSS), state which law you will use, then solve completely.

1. $A = 44$, $C = 81$, $c = 20$.

2. $a = 13$, $c = 17$, $B = 72$.

3. $a = 9$, $b = 12$, $c = 16$.

4. $a = 11$, $b = 16$, $A = 42$.

5. $B = 55$, $C = 62$, $b = 25$.

6. $b = 8,\ c = 13,\ A = 95$.

7. (Real Life) Two roads leave an intersection at an angle of 48. A surveyor walks 30 m along one road and 25 m along the other. What is the straight-line distance between the two endpoints? Then find the other two angles of the triangle formed.

8. (Challenge) In triangle ABC, $a = 7$, $b = 9$, $c = 13$. Find all three angles using the Cosine Law, then verify your answers are correct by confirming the three Sine Law ratios $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$, and $\dfrac{c}{\sin C}$ are all equal.

Answer Key

1. $A = 44$, $C = 81$, $c = 20$. *Configuration: AAS. Use Sine Law.*

Step 1: Find B.

$$B = 180 - 44 - 81 = 55$$

Step 2: Find a.

$$a = \frac{20 \times \sin 44}{\sin 81} = \frac{20 \times 0.6947}{0.9877} \approx 14.07$$

Step 3: Find b.

$$b = \frac{20 \times \sin 55}{\sin 81} = \frac{20 \times 0.8192}{0.9877} \approx 16.59$$

$$\boxed{B = 55, \quad a \approx 14.07, \quad b \approx 16.59}$$

2. $a = 13$, $c = 17$, $B = 72$. *Configuration: SAS. Use Cosine Law, then Sine Law.*

Step 1: Find b using the Cosine Law.

$$b^2 = 13^2 + 17^2 - 2(13)(17)\cos 72 = 169 + 289 - 442 \times 0.3090 = 458 - 136.58 = 321.42$$

$$b = \sqrt{321.42} \approx 17.93$$

Step 2: Use the Sine Law to find A.

$$\sin A = \frac{13 \times \sin 72}{17.93} = \frac{13 \times 0.9511}{17.93} = 0.6896$$

$$A = \sin^{-1}(0.6896) \approx 43.60$$

Step 3: Find C.

$$C = 180 - 72 - 43.60 = 64.40$$

$$\boxed{b \approx 17.93, \quad A \approx 43.60, \quad C \approx 64.40}$$

3. $a = 9$, $b = 12$, $c = 16$. *Configuration: SSS. Use Cosine Law, then Sine Law.*

Step 1: Find C (largest side $c = 16$).

$$\cos C = \frac{9^2 + 12^2 - 16^2}{2(9)(12)} = \frac{81 + 144 - 256}{216} = \frac{-31}{216} = -0.1435$$

$$C = \cos^{-1}(-0.1435) \approx 98.25$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{12 \times \sin 98.25}{16} = \frac{12 \times 0.9896}{16} = 0.7422$$

$$B = \sin^{-1}(0.7422) \approx 47.92$$

Step 3: Find A.

$$A = 180 - 98.25 - 47.92 = 33.83$$

$$\boxed{A \approx 33.83, \quad B \approx 47.92, \quad C \approx 98.25}$$

4. $a = 11$, $b = 16$, $A = 42$. *Configuration: SSA. Check ambiguous case.*

Step 1: Compute h.

$$h = 16 \times \sin 42 = 16 \times 0.6691 = 10.706$$

Since $h = 10.706 \leq a = 11 < b = 16$, **two triangles** exist.
Step 2: Find $\sin B$.

$$\sin B = \frac{16 \times \sin 42}{11} = \frac{16 \times 0.6691}{11} = 0.9733$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9733) \approx 76.73, \quad C_1 = 180 - 42 - 76.73 = 61.27$$

$$c_1 = \frac{11 \times \sin 61.27}{\sin 42} = \frac{11 \times 0.8763}{0.6691} \approx 14.42$$

Triangle 2:

$$B_2 = 180 - 76.73 = 103.27, \quad C_2 = 180 - 42 - 103.27 = 34.73$$

$$c_2 = \frac{11 \times \sin 34.73}{\sin 42} = \frac{11 \times 0.5693}{0.6691} \approx 9.36$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 76.73,\ C_1 \approx 61.27,\ c_1 \approx 14.42 \\ \textbf{Triangle 2:} & B_2 \approx 103.27,\ C_2 \approx 34.73,\ c_2 \approx 9.36 \end{array}}$$

5. $B = 55$, $C = 62$, $b = 25$. *Configuration: AAS. Use Sine Law.*

Step 1: Find A.

$$A = 180 - 55 - 62 = 63$$

Step 2: Find a.

$$a = \frac{25 \times \sin 63}{\sin 55} = \frac{25 \times 0.8910}{0.8192} \approx 27.19$$

Step 3: Find c.

$$c = \frac{25 \times \sin 62}{\sin 55} = \frac{25 \times 0.8829}{0.8192} \approx 26.95$$

$$\boxed{A = 63, \quad a \approx 27.19, \quad c \approx 26.95}$$

6. $b = 8$, $c = 13$, $A = 95$. *Configuration: SAS. Use Cosine Law, then Sine Law.*

Step 1: Note $\cos 95 \approx -0.0872$**. Find** a**.**

$$a^2 = 8^2 + 13^2 - 2(8)(13)\cos 95 = 64 + 169 - 208 \times (-0.0872) = 233 + 18.14 = 251.14$$

$$a = \sqrt{251.14} \approx 15.85$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{8 \times \sin 95}{15.85} = \frac{8 \times 0.9962}{15.85} = 0.5028$$

$$B = \sin^{-1}(0.5028) \approx 30.19$$

Step 3: Find C.

$$C = 180 - 95 - 30.19 = 54.81$$

$$\boxed{a \approx 15.85, \quad B \approx 30.19, \quad C \approx 54.81}$$

7. $b = 30$ m, $c = 25$ m, $A = 48$. *Configuration: SAS. Use Cosine Law, then Sine Law.*

Step 1: Find the distance a.

$$a^2 = 30^2 + 25^2 - 2(30)(25)\cos 48 = 900 + 625 - 1500 \times 0.6691 = 1525 - 1003.65 = 521.35$$

$$a = \sqrt{521.35} \approx 22.83 \text{ m}$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{30 \times \sin 48}{22.83} = \frac{30 \times 0.7431}{22.83} = 0.9764$$

$$B = \sin^{-1}(0.9764) \approx 77.54$$

Step 3: Find C.

$$C = 180 - 48 - 77.54 = 54.46$$

$$\boxed{a \approx 22.83 \text{ m}, \quad B \approx 77.54, \quad C \approx 54.46}$$

8. (Challenge) $a = 7$, $b = 9$, $c = 13$. *Configuration: SSS.*

Step 1: Find C (largest side).

$$\cos C = \frac{7^2 + 9^2 - 13^2}{2(7)(9)} = \frac{49 + 81 - 169}{126} = \frac{-39}{126} = -0.3095$$

$$C = \cos^{-1}(-0.3095) \approx 108.03$$

Step 2: Find B.

$$\cos B = \frac{7^2 + 13^2 - 9^2}{2(7)(13)} = \frac{49 + 169 - 81}{182} = \frac{137}{182} = 0.7527$$

$$B = \cos^{-1}(0.7527) \approx 41.17$$

Step 3: Find A.

$$A = 180 - 108.03 - 41.17 = 30.80$$

Step 4: Verify using Sine Law ratios.

$$\frac{a}{\sin A} = \frac{7}{\sin 30.80} = \frac{7}{0.5120} \approx 13.67$$

$$\frac{b}{\sin B} = \frac{9}{\sin 41.17} = \frac{9}{0.6583} \approx 13.67$$

$$\frac{c}{\sin C} = \frac{13}{\sin 108.03} = \frac{13}{0.9511} \approx 13.67$$

All three ratios are equal. ✓

$$\boxed{A \approx 30.80, \quad B \approx 41.17, \quad C \approx 108.03. \quad \text{Sine Law ratios all} \approx 13.67. \checkmark}$$

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