The Cosine Law: Finding a Side
ApexMath

Notes

When the Sine Law Does Not Work

The Sine Law requires a matching pair — one known side and its opposite angle. But sometimes a problem gives you two sides and the angle *between* them (SAS), with no matching pair available. In that case, the Sine Law cannot be used. The **Cosine Law** is the tool that handles this situation.

The Triangle Configurations

As a reminder, the acronyms used to describe what information is given:

- **SAS — Side, Angle, Side:** Two sides and the angle *between* them are known. The angle is sandwiched between the two known sides. Use the Cosine Law to find the missing side.
- **SSS — Side, Side, Side:** All three sides are known and you need to find an angle. The Cosine Law handles this too — it is covered in Lesson 5.

The Cosine Law — Finding a Side

In any triangle with sides a, b, c opposite angles A, B, C:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

This formula finds the side *opposite* the known angle. The two known sides are b and c, and the known angle A is between them.

The formula can also be written for the other sides:

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \text{and} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

All three versions say the same thing: pick the angle you know, and the side opposite it is what you are solving for.

Connection to the Pythagorean Theorem

When the known angle is $A = 90$, then $\cos 90 = 0$, and the formula becomes:

$$a^2 = b^2 + c^2 - 2bc(0) = b^2 + c^2$$

This is exactly the Pythagorean theorem. The Cosine Law is a *generalization* of it — it works for any angle, not just 90.

Important Note for Obtuse Angles

When the known angle A is obtuse (greater than 90), its cosine is *negative*. This means the term $-2bc\cos A$ becomes *positive*, making a^2 larger than $b^2 + c^2$. This makes sense geometrically: the side opposite an obtuse angle is always the longest side of the triangle.

Step-by-Step Process

- **Step 1:** Confirm you have SAS — two sides and the angle between them.
- **Step 2:** Label the triangle so the known angle is A and the two known sides are b and c.
- **Step 3:** Substitute into $a^2 = b^2 + c^2 - 2bc\cos A$.
- **Step 4:** Evaluate the right side fully, then take the square root to find a.
- **Step 5:** If needed, use the Sine Law to find remaining angles.

Pro Tip

- The formula solves for the side *opposite* the known angle. Make sure you label the unknown side as a and the known angle as A before substituting. If the problem labels things differently, relabel them for your working.
- When the angle is obtuse, $\cos A$ is negative. Subtracting a negative number means you are adding. Write this step out explicitly to avoid sign errors: $-2bc\cos(120) = -2bc(-0.5) = +bc$.
- Do not take the square root until the entire right-hand side is fully simplified. Compute $b^2 + c^2 - 2bc\cos A$ as one number first, then apply $\sqrt{\ }$.
- After finding the missing side, you can use the Sine Law to find the remaining angles. This two-step approach — Cosine Law first, then Sine Law — is the standard method for completing a SAS triangle.
- Always verify using the angle sum $A + B + C = 180$ at the end of any multi-step problem.

Examples

Example 1 (SAS — Basic): In triangle ABC, $b = 8$, $c = 11$, and $A = 60$. Find side a.

Step 1: Confirm SAS. We have two sides (b and c) and the angle between them (A). Apply the Cosine Law.

Step 2: Substitute.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = 8^2 + 11^2 - 2(8)(11)\cos 60$$

Step 3: Evaluate each part.

$$a^2 = 64 + 121 - 176 \times 0.5000$$

$$a^2 = 64 + 121 - 88 = 97$$

Step 4: Take the square root.

$$a = \sqrt{97}$$

$$\boxed{a \approx 9.85}$$

Example 2 (SAS — Obtuse Angle): In triangle ABC, $b = 7$, $c = 10$, and $A = 120$. Find side a.

Step 1: Note that $A = 120$ is obtuse, so $\cos 120 = -0.5$.
Step 2: Substitute.

$$a^2 = 7^2 + 10^2 - 2(7)(10)\cos 120$$

Step 3: Evaluate. Subtracting a negative becomes adding.

$$a^2 = 49 + 100 - 140 \times (-0.5000)$$

$$a^2 = 49 + 100 + 70 = 219$$

Step 4: Take the square root.

$$a = \sqrt{219}$$

$$\boxed{a \approx 14.80}$$

Example 3 (SAS — Complete the Triangle): In triangle ABC, $b = 9$, $c = 13$, and $A = 48$. Find side a, then find angles B and C.

Step 1: Find a using the Cosine Law.

$$a^2 = 9^2 + 13^2 - 2(9)(13)\cos 48$$

$$a^2 = 81 + 169 - 234 \times 0.6691$$

$$a^2 = 81 + 169 - 156.57 = 93.43$$

$$a = \sqrt{93.43} \approx 9.67$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{9 \times \sin 48}{9.67} = \frac{9 \times 0.7431}{9.67} = 0.6918$$

$$B = \sin^{-1}(0.6918) \approx 43.79$$

Step 3: Find C using the angle sum.

$$C = 180 - 48 - 43.79 = 88.21$$

$$\boxed{a \approx 9.67, \quad B \approx 43.79, \quad C \approx 88.21}$$

Example 4 (Connection to Pythagoras): In triangle ABC, $b = 3$, $c = 4$, and $A = 90$. Use the Cosine Law to find a and verify it matches the Pythagorean theorem.

Step 1: Substitute. Note $\cos 90 = 0$.

$$a^2 = 3^2 + 4^2 - 2(3)(4)\cos 90$$

$$a^2 = 9 + 16 - 24 \times 0 = 25$$

Step 2: Take the square root.

$$a = \sqrt{25} = 5$$

This is identical to $a = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ from the Pythagorean theorem, confirming the connection.

$$\boxed{a = 5}$$

Example 5 (Real Life — Distance Between Two Ships): Two ships leave the same port. Ship A travels 30 km and Ship B travels 40 km. The angle between their paths is 70. How far apart are the two ships?

Step 1: Sketch the triangle. The port is the vertex with the known angle. The two known sides are 30 km and 40 km. The unknown side is the distance between the ships.

$$b = 30, \qquad c = 40, \qquad A = 70$$

Step 2: Substitute into the Cosine Law.

$$a^2 = 30^2 + 40^2 - 2(30)(40)\cos 70$$

$$a^2 = 900 + 1600 - 2400 \times 0.3420$$

$$a^2 = 900 + 1600 - 820.85 = 1679.15$$

Step 3: Take the square root.

$$a = \sqrt{1679.15}$$

$$\boxed{a \approx 40.98 \text{ km}}$$

Common Mistakes

- **Using the Cosine Law when the Sine Law would work.** Always check first whether you have a matching side-angle pair. If you do, use the Sine Law — it involves fewer steps. Only reach for the Cosine Law when no matching pair exists.
- **Forgetting the negative sign when $\cos A$ is negative.** For obtuse angles, $\cos A < 0$, so $-2bc\cos A$ becomes a positive number. Writing it as negative is a sign error that makes a^2 too small.
- **Taking the square root too early.** You must fully evaluate $b^2 + c^2 - 2bc\cos A$ as a single number before applying the square root. Taking $\sqrt{b^2 + c^2}$ first and then subtracting is incorrect.
- **Solving for the wrong side.** The formula $a^2 = b^2 + c^2 - 2bc\cos A$ finds the side opposite A. If you want the side opposite B, you must use $b^2 = a^2 + c^2 - 2ac\cos B$.
- **Rounding intermediate values.** Keep at least four decimal places in $\cos A$ and in a^2 before taking the square root. Rounding a^2 to a whole number first introduces error into the final answer.

Worksheet

1. In triangle ABC, $b = 6$, $c = 9$, $A = 55$. Find side a.

2. In triangle ABC, $b = 12$, $c = 15$, $A = 100$. Find side a.

3. In triangle ABC, $b = 5$, $c = 8$, $A = 45$. Find side a.

4. In triangle ABC, $b = 10$, $c = 10$, $A = 72$. Find side a.

5. In triangle ABC, $b = 14$, $c = 20$, $A = 35$. Find side a.

6. In triangle ABC, $b = 7$, $c = 11$, $A = 130$. Find side a.

7. **(Real Life)** Two hiking trails leave from the same trailhead. Trail A runs for 5 km and Trail B runs for 8 km. The angle between them at the trailhead is 64. How far apart are the ends of the two trails?

8. **(Challenge)** In triangle ABC, $b = 9$, $c = 9$, and $A = 40$. Find side a, then use the Sine Law to find angles B and C. What do you notice about B and C, and why does that make sense?

Answer Key

1. $b = 6,\ c = 9,\ A = 55.$

Step 1: Substitute.

$$a^2 = 6^2 + 9^2 - 2(6)(9)\cos 55$$

Step 2: Evaluate.

$$a^2 = 36 + 81 - 108 \times 0.5736 = 117 - 61.95 = 55.05$$

Step 3: Square root.

$$a = \sqrt{55.05}$$

$$\boxed{a \approx 7.42}$$

2. $b = 12,\ c = 15,\ A = 100.$

Step 1: Note $\cos 100 \approx -0.1736$.

$$a^2 = 12^2 + 15^2 - 2(12)(15)\cos 100$$

Step 2: Evaluate. Subtracting a negative becomes adding.

$$a^2 = 144 + 225 - 360 \times (-0.1736)$$

$$a^2 = 144 + 225 + 62.51 = 431.51$$

Step 3: Square root.

$$a = \sqrt{431.51}$$

$$\boxed{a \approx 20.77}$$

3. $b = 5,\ c = 8,\ A = 45.$

Step 1: Substitute.

$$a^2 = 5^2 + 8^2 - 2(5)(8)\cos 45$$

Step 2: Evaluate.

$$a^2 = 25 + 64 - 80 \times 0.7071 = 89 - 56.57 = 32.43$$

Step 3: Square root.

$$a = \sqrt{32.43}$$

$$\boxed{a \approx 5.69}$$

4. $b = 10,\ c = 10,\ A = 72.$

Step 1: Substitute.

$$a^2 = 10^2 + 10^2 - 2(10)(10)\cos 72$$

Step 2: Evaluate.

$$a^2 = 100 + 100 - 200 \times 0.3090 = 200 - 61.80 = 138.20$$

Step 3: Square root.

$$a = \sqrt{138.20}$$

$$\boxed{a \approx 11.76}$$

5. $b = 14,\ c = 20,\ A = 35.$

Step 1: Substitute.

$$a^2 = 14^2 + 20^2 - 2(14)(20)\cos 35$$

Step 2: Evaluate.

$$a^2 = 196 + 400 - 560 \times 0.8192 = 596 - 458.75 = 137.25$$

Step 3: Square root.

$$a = \sqrt{137.25}$$

$$\boxed{a \approx 11.72}$$

6. $b = 7,\ c = 11,\ A = 130.$

Step 1: Note $\cos 130 \approx -0.6428$.

$$a^2 = 7^2 + 11^2 - 2(7)(11)\cos 130$$

Step 2: Evaluate.

$$a^2 = 49 + 121 - 154 \times (-0.6428)$$

$$a^2 = 49 + 121 + 98.99 = 268.99$$

Step 3: Square root.

$$a = \sqrt{268.99}$$

$$\boxed{a \approx 16.40}$$

7. $b = 5$ km, $c = 8$ km, $A = 64$.

Step 1: Substitute.

$$a^2 = 5^2 + 8^2 - 2(5)(8)\cos 64$$

Step 2: Evaluate.

$$a^2 = 25 + 64 - 80 \times 0.4384 = 89 - 35.07 = 53.93$$

Step 3: Square root.

$$a = \sqrt{53.93}$$

$$\boxed{a \approx 7.34 \text{ km}}$$

8. (Challenge) $b = 9$, $c = 9$, $A = 40$.

Step 1: Find a using the Cosine Law.

$$a^2 = 9^2 + 9^2 - 2(9)(9)\cos 40$$

$$a^2 = 81 + 81 - 162 \times 0.7660 = 162 - 124.10 = 37.90$$

$$a = \sqrt{37.90} \approx 6.16$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{9 \times \sin 40}{6.16} = \frac{9 \times 0.6428}{6.16} = 0.9394$$

$$B = \sin^{-1}(0.9394) \approx 70.00$$

Step 3: Find C.

$$C = 180 - 40 - 70 = 70$$

Step 4: Observation. $B = C = 70$. This makes sense because $b = c = 9$ — the triangle is isosceles, and in any isosceles triangle, the two equal sides are always opposite equal angles.

$$\boxed{a \approx 6.16, \quad B = C = 70. \quad \text{The triangle is isosceles: equal sides produce equal angles.}}$$

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