The Cosine Law: Finding an Angle
ApexMath

Notes

The Other Use of the Cosine Law

In Lesson 4, the Cosine Law found a missing side when two sides and the included angle were given (SAS). Now we use the same formula in reverse: when all three sides are known (SSS), we can rearrange the Cosine Law to find any missing angle.

The Configuration: SSS

- **SSS — Side, Side, Side:** All three side lengths are known and you need to find one or more angles. There is no matching side-angle pair, so the Sine Law cannot be used to start. The Cosine Law must come first.

The Rearranged Cosine Law

Starting from $a^2 = b^2 + c^2 - 2bc\cos A$, isolate $\cos A$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Then apply the inverse cosine to find the angle:

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

The same rearrangement works for the other angles:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \text{and} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Reading the Sign of $\cos A$

The sign of the result tells you immediately what kind of angle you have:

- $\cos A > 0$: angle A is **acute** (less than 90)
- $\cos A = 0$: angle A is exactly 90
- $\cos A < 0$: angle A is **obtuse** (greater than 90)

This is a useful built-in check. The largest side of any triangle is always opposite the largest angle, so if a side is noticeably longer than the others, expect its opposite angle to be obtuse.

Strategy for Finding All Three Angles

- **Step 1:** Use the Cosine Law to find the largest angle first (the one opposite the longest side). This is important because there can only ever be one obtuse angle in a triangle. Finding it with the Cosine Law removes all ambiguity.
- **Step 2:** Use the Sine Law to find one of the remaining angles (now that you have a matching pair).
- **Step 3:** Use $A + B + C = 180$ to find the last angle.

Pro Tip

- Always find the largest angle first using the Cosine Law. Once you know whether the triangle has an obtuse angle or not, the remaining angles found with the Sine Law are unambiguous — they must both be acute.
- The numerator $b^2 + c^2 - a^2$ can be negative. This is not an error. It simply means the angle is obtuse. Do not discard a negative numerator — divide it by $2bc$ and apply $\cos^{-1}$ as normal. Your calculator will return an angle between 90 and 180.
- Label the sides carefully before substituting. In the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, the side a is the one *opposite* the angle you are finding. The sides b and c are the other two.
- After finding all three angles, always verify: $A + B + C = 180$. If the sum is off by more than 0.1, recheck your calculation.
- For an equilateral triangle (all sides equal), every angle is exactly 60. You can confirm this instantly with the formula as a quick sanity check on your calculator work.

Examples

Example 1 (SSS — Find One Angle): In triangle ABC, $a = 7$, $b = 8$, $c = 10$. Find angle A.

Step 1: Write the rearranged Cosine Law for angle A.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Step 2: Substitute.

$$\cos A = \frac{8^2 + 10^2 - 7^2}{2(8)(10)} = \frac{64 + 100 - 49}{160} = \frac{115}{160} = 0.7188$$

Step 3: Apply $\cos^{-1}$.

$$A = \cos^{-1}(0.7188)$$

$$\boxed{A \approx 44.05}$$

Example 2 (SSS — Find All Three Angles): In triangle ABC, $a = 5$, $b = 7$, $c = 9$. Find all three angles.

Step 1: Find angle A (opposite the shortest side).

$$\cos A = \frac{7^2 + 9^2 - 5^2}{2(7)(9)} = \frac{49 + 81 - 25}{126} = \frac{105}{126} = 0.8333$$

$$A = \cos^{-1}(0.8333) \approx 33.56$$

Step 2: Find angle B.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 9^2 - 7^2}{2(5)(9)} = \frac{25 + 81 - 49}{90} = \frac{57}{90} = 0.6333$$

$$B = \cos^{-1}(0.6333) \approx 50.70$$

Step 3: Find angle C using the angle sum.

$$C = 180 - 33.56 - 50.70 = 95.74$$

Step 4: Verify. $33.56 + 50.70 + 95.74 = 180.00$ ✓

$$\boxed{A \approx 33.56, \quad B \approx 50.70, \quad C \approx 95.74}$$

Example 3 (SSS — Obtuse Angle Result): In triangle ABC, $a = 14$, $b = 8$, $c = 9$. Find angle A.

Step 1: Side $a = 14$ is the longest, so angle A will be the largest. Substitute.

$$\cos A = \frac{8^2 + 9^2 - 14^2}{2(8)(9)} = \frac{64 + 81 - 196}{144} = \frac{-51}{144} = -0.3542$$

Step 2: The numerator is negative, so A is obtuse. Apply $\cos^{-1}$.

$$A = \cos^{-1}(-0.3542)$$

$$\boxed{A \approx 110.74}$$

Example 4 (SSS — Find the Largest Angle First): In triangle ABC, $a = 6$, $b = 8$, $c = 11$. Find all three angles.

Step 1: The longest side is $c = 11$, so find C first.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{6^2 + 8^2 - 11^2}{2(6)(8)} = \frac{36 + 64 - 121}{96} = \frac{-21}{96} = -0.2188$$

$$C = \cos^{-1}(-0.2188) \approx 102.64$$

Step 2: Use the Sine Law to find B.

$$\sin B = \frac{b \sin C}{c} = \frac{8 \times \sin 102.64}{11} = \frac{8 \times 0.9757}{11} = 0.7096$$

$$B = \sin^{-1}(0.7096) \approx 45.21$$

Step 3: Find A.

$$A = 180 - 102.64 - 45.21 = 32.15$$

$$\boxed{C \approx 102.64, \quad B \approx 45.21, \quad A \approx 32.15}$$

Example 5 (Real Life — Triangular Plot of Land): A triangular plot of land has sides of 85 m, 95 m, and 120 m. Find the largest angle of the plot.

Step 1: The largest side is 120 m. Let $c = 120$, $a = 85$, $b = 95$. Find angle C.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{85^2 + 95^2 - 120^2}{2(85)(95)} = \frac{7225 + 9025 - 14400}{16150} = \frac{1850}{16150} = 0.1146$$

Step 2: Apply $\cos^{-1}$.

$$C = \cos^{-1}(0.1146)$$

$$\boxed{C \approx 83.42}$$

Common Mistakes

- **Putting the wrong side in the numerator.** In $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, the side being subtracted in the numerator (a^2) must be the side *opposite* the angle you are finding. Putting b^2 or c^2 in that position gives the wrong angle entirely.

- **Discarding a negative numerator.** A negative value of $b^2 + c^2 - a^2$ is not an error — it means the angle is obtuse. Divide it as normal and apply $\cos^{-1}$. Your calculator will correctly return an angle greater than 90.

- **Using the Sine Law first for SSS.** Without a known angle, there is no matching side-angle pair to start the Sine Law. Always begin SSS problems with the Cosine Law.

- **Not verifying the angle sum.** After finding all three angles, always check $A + B + C = 180$. A small rounding discrepancy (± 0.1) is acceptable; anything larger indicates a calculation error.
- **Squaring incorrectly.** A very common arithmetic error is computing a^2 wrong, especially for two-digit numbers. Write out each square explicitly: $14^2 = 196$, not 168 or 196 from memory. Double-check each one.

Worksheet

1. In triangle ABC, $a = 9$, $b = 11$, $c = 14$. Find angle A.

2. In triangle ABC, $a = 6$, $b = 7$, $c = 8$. Find all three angles.

3. In triangle ABC, $a = 15$, $b = 10$, $c = 12$. Find angle A (the largest angle).

4. In triangle ABC, $a = 4$, $b = 5$, $c = 7$. Find all three angles.

5. In triangle ABC, $a = 13$, $b = 13$, $c = 10$. Find all three angles. What do you notice?

6. In triangle ABC, $a = 20$, $b = 14$, $c = 15$. Find the largest angle.

7. **(Real Life)** A surveyor measures a triangular piece of land with sides 200 m, 150 m, and 175 m. Find all three angles of the triangle.

8. **(Challenge)** In triangle ABC, $a = b = c = 10$. Use the Cosine Law to find all three angles. What type of triangle is this, and does the result match what you would expect?

Answer Key

1. $a = 9$, $b = 11$, $c = 14$. Find A.

Step 1: Substitute.

$$\cos A = \frac{11^2 + 14^2 - 9^2}{2(11)(14)} = \frac{121 + 196 - 81}{308} = \frac{236}{308} = 0.7662$$

Step 2: Apply $\cos^{-1}$.

$$A = \cos^{-1}(0.7662)$$

$$\boxed{A \approx 39.98}$$

2. $a = 6$, $b = 7$, $c = 8$. Find all three angles.

Step 1: Find A.

$$\cos A = \frac{7^2 + 8^2 - 6^2}{2(7)(8)} = \frac{49 + 64 - 36}{112} = \frac{77}{112} = 0.6875$$

$$A = \cos^{-1}(0.6875) \approx 46.57$$

Step 2: Find B.

$$\cos B = \frac{6^2 + 8^2 - 7^2}{2(6)(8)} = \frac{36 + 64 - 49}{96} = \frac{51}{96} = 0.5313$$

$$B = \cos^{-1}(0.5313) \approx 57.91$$

Step 3: Find C.

$$C = 180 - 46.57 - 57.91 = 75.52$$

$$\boxed{A \approx 46.57, \quad B \approx 57.91, \quad C \approx 75.52}$$

3. $a = 15$, $b = 10$, $c = 12$. Find A (largest angle).

Step 1: Side $a = 15$ is the longest. Find A.

$$\cos A = \frac{10^2 + 12^2 - 15^2}{2(10)(12)} = \frac{100 + 144 - 225}{240} = \frac{19}{240} = 0.0792$$

Step 2: Apply $\cos^{-1}$.

$$A = \cos^{-1}(0.0792)$$

$$\boxed{A \approx 85.46}$$

4. $a = 4$, $b = 5$, $c = 7$. Find all three angles.

Step 1: Find C first (largest side $c = 7$).

$$\cos C = \frac{4^2 + 5^2 - 7^2}{2(4)(5)} = \frac{16 + 25 - 49}{40} = \frac{-8}{40} = -0.2000$$

$$C = \cos^{-1}(-0.2000) \approx 101.54$$

Step 2: Find A.

$$\cos A = \frac{5^2 + 7^2 - 4^2}{2(5)(7)} = \frac{25 + 49 - 16}{70} = \frac{58}{70} = 0.8286$$

$$A = \cos^{-1}(0.8286) \approx 34.05$$

Step 3: Find B.

$$B = 180 - 101.54 - 34.05 = 44.41$$

$$\boxed{A \approx 34.05, \quad B \approx 44.41, \quad C \approx 101.54}$$

5. $a = 13$, $b = 13$, $c = 10$. Find all three angles.

Step 1: Find C (opposite the shortest side $c = 10$).

$$\cos C = \frac{13^2 + 13^2 - 10^2}{2(13)(13)} = \frac{169 + 169 - 100}{338} = \frac{238}{338} = 0.7041$$

$$C = \cos^{-1}(0.7041) \approx 45.24$$

Step 2: Find A (or B — they are equal since $a = b$).

$$\cos A = \frac{13^2 + 10^2 - 13^2}{2(13)(10)} = \frac{169 + 100 - 169}{260} = \frac{100}{260} = 0.3846$$

$$A = \cos^{-1}(0.3846) \approx 67.38$$

Step 3: Since $a = b$, angle $B = A = 67.38$. Verify.

$$67.38 + 67.38 + 45.24 = 180.00\checkmark$$

The triangle is isosceles ($a = b$), so $A = B$ as expected.

$$\boxed{A = B \approx 67.38, \quad C \approx 45.24}$$

6. $a = 20$, $b = 14$, $c = 15$. Find the largest angle.

Step 1: Largest side is $a = 20$. Find A.

$$\cos A = \frac{14^2 + 15^2 - 20^2}{2(14)(15)} = \frac{196 + 225 - 400}{420} = \frac{21}{420} = 0.0500$$

$$A = \cos^{-1}(0.0500)$$

$$\boxed{A \approx 87.13}$$

7. Sides 200 m, 150 m, 175 m. Let $a = 150$, $b = 175$, $c = 200$.

Step 1: Find C (largest side $c = 200$).

$$\cos C = \frac{150^2 + 175^2 - 200^2}{2(150)(175)} = \frac{22500 + 30625 - 40000}{52500} = \frac{13125}{52500} = 0.2500$$

$$C = \cos^{-1}(0.2500) \approx 75.52$$

Step 2: Find B.

$$\cos B = \frac{150^2 + 200^2 - 175^2}{2(150)(200)} = \frac{22500 + 40000 - 30625}{60000} = \frac{31875}{60000} = 0.5313$$

$$B = \cos^{-1}(0.5313) \approx 57.91$$

Step 3: Find A.

$$A = 180 - 75.52 - 57.91 = 46.57$$

$$\boxed{A \approx 46.57, \quad B \approx 57.91, \quad C \approx 75.52}$$

8. (Challenge) $a = b = c = 10$.

Step 1: Find angle A using the Cosine Law.

$$\cos A = \frac{10^2 + 10^2 - 10^2}{2(10)(10)} = \frac{100 + 100 - 100}{200} = \frac{100}{200} = 0.5000$$

$$A = \cos^{-1}(0.5000) = 60$$

Step 2: By symmetry, $B = C = 60$ as well.

$$60 + 60 + 60 = 180\checkmark$$

This is an **equilateral triangle**. When all three sides are equal, all three angles are equal, and since they must sum to 180, each must be exactly 60. The Cosine Law confirms this perfectly.

$$\boxed{A = B = C = 60. \quad \text{This is an equilateral triangle.}}$$

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