The Sine Law
ApexMath

Notes

Why We Need a New Tool

SOH-CAH-TOA only works in right triangles. Most triangles in the real world do not have a 90 angle. The **Sine Law** is the first of two tools that work in *any* triangle — right, acute, or obtuse.

The Sine Law

In any triangle with sides a, b, c opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This can also be written upside down, which is more convenient when solving for an angle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Both forms say the same thing: each side divided by the sine of its opposite angle gives the same number for every side in the triangle.

When to Use the Sine Law

The Sine Law works whenever you have a **matching pair** — one known side and its opposite angle. You then need one more piece of information:

- **AAS** (two angles and any side): use the Sine Law to find a missing side. Find the third angle first using $A + B + C = 180$.
- **ASA** (two angles and the included side): same process as AAS — find the third angle first, then apply the law.
- **SSA** (two sides and an angle opposite one of them): use the Sine Law to find a missing angle. Be careful — this case can sometimes produce two answers (the ambiguous case, covered in Lesson 3). In this lesson, all SSA problems are set up to give exactly one answer.

Step-by-Step Process

- **Step 1:** Label the triangle. Assign a, b, c to the sides and A, B, C to the opposite angles.
- **Step 2:** Identify what is given and what is unknown.
- **Step 3:** If two angles are given, find the third using $A + B + C = 180$.

- **Step 4:** Set up the Sine Law using the pair that has both values known, and the pair containing the unknown.
- **Step 5:** Cross-multiply and solve.

Pro Tip

- Only use two ratios at a time. The Sine Law has three ratios, but you only ever set two of them equal to each other. Pick the pair that has the unknown and the pair that is fully known.
- Always find the third angle first when two angles are given. This takes one step and immediately gives you a complete set of angles to work with.
- When solving for an angle, use the upside-down form $\frac{\sin A}{a} = \frac{\sin B}{b}$ — it avoids having to divide by a sine value and is less error-prone.
- After finding an angle with $\sin^{-1}$, always ask: could there be a second answer? A supplementary angle $(180 - \theta)$ also has the same sine value. In this lesson all problems have exactly one solution, but developing this habit now will prepare you for the ambiguous case in Lesson 3.
- Check your answer using the angle sum: after finding all three angles, confirm they add to exactly 180.

Examples

Example 1 (AAS — Find a Side): In triangle ABC, $A = 40$, $B = 75$, and $b = 12$. Find side a.

Step 1: Find the third angle.

$$C = 180 - 40 - 75 = 65$$

Step 2: Identify the known pair and the unknown. We know b and B, and want a.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Step 3: Substitute.

$$\frac{a}{\sin 40} = \frac{12}{\sin 75}$$

Step 4: Solve for a by multiplying both sides by $\sin 40$.

$$a = \frac{12 \times \sin 40}{\sin 75} = \frac{12 \times 0.6428}{0.9659}$$

$$\boxed{a \approx 7.99}$$

Example 2 (ASA — Find a Side): In triangle ABC, $A = 55$, $C = 80$, and $c = 20$. Find side b.

Step 1: Find the third angle.

$$B = 180 - 55 - 80 = 45$$

Step 2: Set up the Sine Law using the known pair (c, C) and the unknown b.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Step 3: Substitute.

$$\frac{b}{\sin 45} = \frac{20}{\sin 80}$$

Step 4: Solve for b.

$$b = \frac{20 \times \sin 45}{\sin 80} = \frac{20 \times 0.7071}{0.9848}$$

$$\boxed{b \approx 14.36}$$

Example 3 (SSA — Find an Angle): In triangle ABC, $a = 10$, $b = 14$, and $A = 30$. Find angle B.

Step 1: Use the upside-down form to solve for $\sin B$.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

Step 2: Substitute.

$$\frac{\sin B}{14} = \frac{\sin 30}{10}$$

Step 3: Multiply both sides by 14.

$$\sin B = \frac{14 \times \sin 30}{10} = \frac{14 \times 0.5}{10} = 0.7$$

Step 4: Apply $\sin^{-1}$ to find B.

$$B = \sin^{-1}(0.7) \approx 44.43$$

Step 5: Find C and verify.

$$C = 180 - 30 - 44.43 = 105.57$$

$$30 + 44.43 + 105.57 = 180\checkmark$$

$$\boxed{B \approx 44.43}$$

Example 4 (SSA — Find an Angle, Then Complete the Triangle): In triangle ABC, $b = 25$, $c = 30$, and $B = 50$. Find angle C, then find angle A and side a.

Step 1: Use the upside-down form to find $\sin C$.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{30} = \frac{\sin 50}{25}$$

$$\sin C = \frac{30 \times \sin 50}{25} = \frac{30 \times 0.7660}{25} = 0.9192$$

Step 2: Find C.

$$C = \sin^{-1}(0.9192) \approx 66.82$$

Step 3: Find A.

$$A = 180 - 50 - 66.82 = 63.18$$

Step 4: Find a using the Sine Law.

$$a = \frac{b \times \sin A}{\sin B} = \frac{25 \times \sin 63.18}{\sin 50} = \frac{25 \times 0.8921}{0.7660}$$

$$\boxed{C \approx 66.82, \quad A \approx 63.18, \quad a \approx 29.13}$$

Example 5 (Real Life — Distance to a Fire): Two park rangers are stationed 15 km apart at points A and B. They both spot a forest fire at point F. The angle at A between the line AB and the fire is 52. The angle at B between the line AB and the fire is 63. How far is the fire from each ranger?

Step 1: Find the third angle at the fire.

$$F = 180 - 52 - 63 = 65$$

Step 2: Label the triangle. Side $AB = 15$ km is opposite the angle at F. Side AF is opposite angle B. Side BF is opposite angle A.

Step 3: Find AF (distance from ranger A to the fire).

$$\frac{AF}{\sin B} = \frac{AB}{\sin F}$$

$$AF = \frac{15 \times \sin 63}{\sin 65} = \frac{15 \times 0.8910}{0.9063}$$

$$AF \approx 14.75 \text{ km}$$

Step 4: Find BF (distance from ranger B to the fire).

$$BF = \frac{15 \times \sin 52}{\sin 65} = \frac{15 \times 0.7880}{0.9063}$$

$$\boxed{AF \approx 14.75 \text{ km}, \quad BF \approx 13.04 \text{ km}}$$

Common Mistakes

- **Using a side and a non-opposite angle in the same ratio.** The Sine Law requires that each side is paired with the angle *directly opposite* it. Double-check your labelling before setting up any ratio.
- **Forgetting to find the third angle first.** When two angles are given, the third is free — one subtraction gives you a complete angle set. Skipping this step means you cannot set up the second ratio.
- **Cross-multiplying incorrectly.** From $\frac{a}{\sin A} = \frac{b}{\sin B}$, multiplying both sides by $\sin A$ gives $a = \frac{b \sin A}{\sin B}$. A common error is writing $a = b \sin A \times \sin B$ (multiplying instead of dividing).
- **Rounding sine values too early.** Keep at least four decimal places in intermediate steps. Rounding to two decimal places mid-calculation causes the final answer to drift.
- **Not checking the angle sum at the end.** Always verify that your three angles add to 180. If they do not, you have made an error somewhere.

Worksheet

1. In triangle ABC, $A = 35$, $B = 85$, and $a = 9$. Find sides b and c.

2. In triangle ABC, $B = 62$, $C = 48$, and $c = 17$. Find sides a and b.

3. In triangle ABC, $a = 8$, $b = 11$, and $A = 35$. Find angle B and then find side c.

4. In triangle ABC, $A = 70$, $B = 58$, and $b = 22$. Find sides a and c.

5. In triangle ABC, $b = 16$, $c = 20$, and $B = 47$. Find angle C, then find angle A and side a.

6. In triangle ABC, $A = 42$, $C = 67$, and $c = 30$. Find sides a and b.

7. **(Real Life)** Two boats are anchored 8 km apart at points A and B. They both observe a lighthouse at point L. The angle at A is 73 and the angle at B is 55. How far is each boat from the lighthouse?

8. **(Challenge)** The three angles of a triangle are in the ratio $1 : 2 : 3$. The shortest side has length 10. Use the Sine Law to find the other two sides. What type of triangle is this?

Answer Key

1. $A = 35,\ B = 85,\ a = 9.$

Step 1: Find C.

$$C = 180 - 35 - 85 = 60$$

Step 2: Find b.

$$b = \frac{a \times \sin B}{\sin A} = \frac{9 \times \sin 85}{\sin 35} = \frac{9 \times 0.9962}{0.5736}$$

$$b \approx 15.63$$

Step 3: Find c.

$$c = \frac{9 \times \sin 60}{\sin 35} = \frac{9 \times 0.8660}{0.5736}$$

$$\boxed{b \approx 15.63, \quad c \approx 13.59}$$

2. $B = 62,\ C = 48,\ c = 17.$

Step 1: Find A.

$$A = 180 - 62 - 48 = 70$$

Step 2: Find b.

$$b = \frac{17 \times \sin 62}{\sin 48} = \frac{17 \times 0.8829}{0.7431} \approx 20.20$$

Step 3: Find a.

$$a = \frac{17 \times \sin 70}{\sin 48} = \frac{17 \times 0.9397}{0.7431} \approx 21.50$$

$$\boxed{A = 70, \quad b \approx 20.20, \quad a \approx 21.50}$$

3. $a = 8,\ b = 11,\ A = 35.$

Step 1: Find $\sin B$.

$$\sin B = \frac{b \times \sin A}{a} = \frac{11 \times \sin 35}{8} = \frac{11 \times 0.5736}{8} = 0.7888$$

Step 2: Find B.

$$B = \sin^{-1}(0.7888) \approx 52.06$$

Step 3: Find C.

$$C = 180 - 35 - 52.06 = 92.94$$

Step 4: Find c.

$$c = \frac{8 \times \sin 92.94}{\sin 35} = \frac{8 \times 0.9987}{0.5736} \approx 13.93$$

$$\boxed{B \approx 52.06, \quad c \approx 13.93}$$

4. $A = 70,\ B = 58,\ b = 22.$

Step 1: Find C.

$$C = 180 - 70 - 58 = 52$$

Step 2: Find a.

$$a = \frac{22 \times \sin 70}{\sin 58} = \frac{22 \times 0.9397}{0.8480} \approx 24.38$$

Step 3: Find c.

$$c = \frac{22 \times \sin 52}{\sin 58} = \frac{22 \times 0.7880}{0.8480} \approx 20.44$$

$$\boxed{a \approx 24.38, \quad c \approx 20.44}$$

5. $b = 16,\ c = 20,\ B = 47.$

Step 1: Find $\sin C$.

$$\sin C = \frac{20 \times \sin 47}{16} = \frac{20 \times 0.7314}{16} = 0.9142$$

Step 2: Find C.

$$C = \sin^{-1}(0.9142) \approx 66.09$$

Step 3: Find A.

$$A = 180 - 47 - 66.09 = 66.91$$

Step 4: Find a.

$$a = \frac{16 \times \sin 66.91}{\sin 47} = \frac{16 \times 0.9196}{0.7314} \approx 20.12$$

$$\boxed{C \approx 66.09, \quad A \approx 66.91, \quad a \approx 20.12}$$

6. $A = 42,\ C = 67,\ c = 30.$

Step 1: Find B.

$$B = 180 - 42 - 67 = 71$$

Step 2: Find a.

$$a = \frac{30 \times \sin 42}{\sin 67} = \frac{30 \times 0.6691}{0.9205} \approx 21.81$$

Step 3: Find b.

$$b = \frac{30 \times \sin 71}{\sin 67} = \frac{30 \times 0.9455}{0.9205} \approx 30.82$$

$$\boxed{B = 71, \quad a \approx 21.81, \quad b \approx 30.82}$$

7. Two boats 8 km apart. Angle at $A = 73$, angle at $B = 55$.

Step 1: Find the angle at the lighthouse.

$$L = 180 - 73 - 55 = 52$$

Step 2: Find AL (distance from boat A to lighthouse).

$$AL = \frac{8 \times \sin 55}{\sin 52} = \frac{8 \times 0.8192}{0.7880} \approx 8.32 \text{ km}$$

Step 3: Find BL (distance from boat B to lighthouse).

$$BL = \frac{8 \times \sin 73}{\sin 52} = \frac{8 \times 0.9563}{0.7880} \approx 9.71 \text{ km}$$

$$\boxed{AL \approx 8.32 \text{ km}, \quad BL \approx 9.71 \text{ km}}$$

8. (Challenge) Angles in ratio 1 : 2 : 3 with shortest side = 10.

Step 1: Find the three angles.

$$x + 2x + 3x = 180 \qquad \Longrightarrow \qquad 6x = 180 \qquad \Longrightarrow \qquad x = 30$$

$$A = 30, \quad B = 60, \quad C = 90$$

Step 2: The shortest side $a = 10$ is opposite the smallest angle $A = 30$. Apply the Sine Law.

$$b = \frac{10 \times \sin 60}{\sin 30} = \frac{10 \times \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 10\sqrt{3}$$

$$c = \frac{10 \times \sin 90}{\sin 30} = \frac{10 \times 1}{\frac{1}{2}} = 20$$

Step 3: Identify the triangle type.

Since $C = 90$, this is a **right triangle** — specifically a 30-60-90 triangle, confirming the result from Lesson 1.

$$\boxed{b = 10\sqrt{3} \approx 17.32, \quad c = 20. \quad \text{This is a 30-60-90 right triangle.}}$$

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