The Ambiguous Case (SSA)
ApexMath

Notes

Understanding the Acronyms: AAS, ASA, and SSA

In triangle problems, each letter stands for either a **S**ide or an **A**ngle. The three letters describe what information you are given, in the order it appears around the triangle.

- **AAS — Angle, Angle, Side:** You know two angles and a side that is *not* between them. For example: angle A, angle B, and side a, where a is opposite A rather than sitting between the two angles. This always produces exactly one triangle.
- **ASA — Angle, Side, Angle:** You know two angles and the side that is *between* them. For example: angle A, side b, and angle B, where side b sits directly between A and B. This also always produces exactly one triangle.
- **SSA — Side, Side, Angle:** You know two sides and an angle that is *opposite* one of them — not between them. For example: side a, side b, and angle A, where A is opposite a. Because the known angle is not locked between the two sides, the triangle can swing into more than one valid position. This is why SSA is the only case that can produce zero, one, *or* two triangles.

The key distinction between AAS and ASA is simply whether the known side is between the two known angles (ASA) or not (AAS). Both are straightforward. SSA is the one that requires extra care.

What Makes SSA Different

In Lesson 2, the Sine Law was used to find a missing side or angle. When two angles and a side are given (AAS or ASA), there is always exactly one triangle. But when two sides and a non-included angle are given (SSA — two sides and the angle *opposite* one of them), something unusual can happen: there may be **zero, one, or two** valid triangles. This is called the **ambiguous case**.

Setup and Vocabulary

Given: angle A, the side adjacent to A called b, and the side opposite A called a. Before solving anything, compute the **height**:

$$h = b \sin A$$

The height h is the minimum length a would need to be to just reach the base and form a triangle. Comparing a to h and to b tells you exactly how many triangles exist.

Case Analysis — Acute Angle A

When angle A is acute (less than 90):

- $a < h$: Side a is too short to reach the base. **No triangle** exists.

- $a = h$: Side a just reaches the base at a right angle. Exactly **one right triangle** exists.
- $h \le a < b$: Side a is long enough to reach the base in **two different positions**. Two triangles exist.
- $a \ge b$: Side a is so long there is only one way it can land. Exactly **one triangle** exists.

Case Analysis — Obtuse Angle A
When angle A is obtuse (greater than 90):

- $a \le b$: No triangle exists. A side opposite an obtuse angle must be the longest side, so a must be greater than b.
- $a > b$: Exactly **one triangle** exists.

Summary Table

Condition	Requirement	Number of Triangles
A acute	$a < h = b \sin A$	0
A acute	$a = h = b \sin A$	1 (right triangle)
A acute	$h \le a < b$	2
A acute	$a \ge b$	1
A obtuse	$a \le b$	0
A obtuse	$a > b$	1

Solving When Two Triangles Exist
When the case analysis shows two triangles, solve for both completely.

- Use the Sine Law to find $\sin B$.
- **Triangle 1:** $B_1 = \sin^{-1}(\sin B)$ (acute angle from calculator)
- **Triangle 2:** $B_2 = 180 - B_1$ (the supplementary angle, which also has the same sine value)
- For each triangle, find the third angle $C = 180 - A - B$, then find side c using the Sine Law.
- Discard any triangle where an angle is negative or zero — that solution is invalid.

Pro Tip

- Always compute $h = b \sin A$ as your very first step in any SSA problem. This single number immediately tells you how many triangles to expect before you do any Sine Law work.
- When you find $B_2 = 180 - B_1$, immediately check that $A + B_2 < 180$ before solving further. If the sum already exceeds 180, that second triangle is impossible and must be discarded.
- In an SSA problem, the given angle A is always opposite the given side a. The side b is always adjacent to A. Getting this pairing backwards is the most common setup error.
- Label both triangles clearly as Triangle 1 and Triangle 2 when two solutions exist. Present them separately, step by step. Mixing the two solutions together in one block of work leads to errors and loses marks.
- After solving, verify each triangle independently using the angle sum $A+B+C = 180$.

Examples

Example 1 (Zero Triangles): In triangle ABC, $A = 35$, $b = 10$, and $a = 5$. Determine how many triangles are possible.

Step 1: Compute the height.

$$h = b \sin A = 10 \times \sin 35 = 10 \times 0.5736 = 5.736$$

Step 2: Compare a to h.

$$a = 5 < h = 5.736$$

Since $a < h$, side a is too short to reach the base.

$$\boxed{\text{No triangle exists.}}$$

Example 2 (One Triangle, $a \geq b$): In triangle ABC, $A = 35$, $b = 10$, and $a = 15$. Solve the triangle.

Step 1: Compute the height.

$$h = 10 \times \sin 35 \approx 5.736$$

Step 2: Compare a to h and b.

$$a = 15 \geq b = 10$$

Since $a \geq b$, exactly one triangle exists.

Step 3: Use the Sine Law to find B.

$$\sin B = \frac{b \sin A}{a} = \frac{10 \times \sin 35}{15} = \frac{10 \times 0.5736}{15} = 0.3824$$

$$B = \sin^{-1}(0.3824) \approx 22.48$$

Step 4: Find C and c.

$$C = 180 - 35 - 22.48 = 122.52$$

$$c = \frac{15 \times \sin 122.52}{\sin 35} = \frac{15 \times 0.8425}{0.5736} \approx 22.05$$

$$\boxed{B \approx 22.48, \quad C \approx 122.52, \quad c \approx 22.05}$$

Example 3 (Two Triangles): In triangle ABC, $A = 40$, $b = 12$, and $a = 9$. Solve all possible triangles.

Step 1: Compute the height.

$$h = 12 \times \sin 40 = 12 \times 0.6428 = 7.714$$

Step 2: Compare a to h and b.

$$h = 7.714 \leq a = 9 < b = 12$$

Two triangles exist.

Step 3: Find $\sin B$ using the Sine Law.

$$\sin B = \frac{12 \times \sin 40}{9} = \frac{12 \times 0.6428}{9} = 0.8571$$

Step 4: Solve Triangle 1 (acute B).

$$B_1 = \sin^{-1}(0.8571) \approx 58.99$$

$$C_1 = 180 - 40 - 58.99 = 81.01$$

$$c_1 = \frac{9 \times \sin 81.01}{\sin 40} = \frac{9 \times 0.9877}{0.6428} \approx 13.83$$

Step 5: Solve Triangle 2 (obtuse B).

$$B_2 = 180 - 58.99 = 121.01$$

$$C_2 = 180 - 40 - 121.01 = 18.99$$

$$c_2 = \frac{9 \times \sin 18.99}{\sin 40} = \frac{9 \times 0.3253}{0.6428} \approx 4.56$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 58.99,\ C_1 \approx 81.01,\ c_1 \approx 13.83 \\ \textbf{Triangle 2:} & B_2 \approx 121.01,\ C_2 \approx 18.99,\ c_2 \approx 4.56 \end{array}}$$

Example 4 (Two Triangles — Fully Solved): In triangle ABC, $A = 45$, $b = 14$, and $a = 11$. Solve all possible triangles completely.

Step 1: Compute the height.

$$h = 14 \times \sin 45 = 14 \times 0.7071 = 9.899$$

Step 2: Compare.

$$h = 9.899 \leq a = 11 < b = 14$$

Two triangles exist.
Step 3: Find $\sin B$.

$$\sin B = \frac{14 \times \sin 45}{11} = \frac{14 \times 0.7071}{11} = 0.9000$$

Step 4: Triangle 1.

$$B_1 = \sin^{-1}(0.9000) \approx 64.16$$

$$C_1 = 180 - 45 - 64.16 = 70.84$$

$$c_1 = \frac{11 \times \sin 70.84}{\sin 45} = \frac{11 \times 0.9450}{0.7071} \approx 14.70$$

Step 5: Triangle 2.

$$B_2 = 180 - 64.16 = 115.84$$

$$C_2 = 180 - 45 - 115.84 = 19.16$$

$$c_2 = \frac{11 \times \sin 19.16}{\sin 45} = \frac{11 \times 0.3283}{0.7071} \approx 5.11$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 64.16,\ C_1 \approx 70.84,\ c_1 \approx 14.70 \\ \textbf{Triangle 2:} & B_2 \approx 115.84,\ C_2 \approx 19.16,\ c_2 \approx 5.11 \end{array}}$$

Example 5 (Obtuse A — Zero Triangles): In triangle ABC, $A = 110$, $b = 12$, and $a = 8$. Determine how many triangles are possible.

Step 1: Identify that A is obtuse. Apply the obtuse rule.

$$a = 8 \leq b = 12$$

When A is obtuse, side a must be strictly greater than b to form any triangle. Since $a \leq b$, no triangle is possible.

$$\boxed{\text{No triangle exists.}}$$

Common Mistakes

- **Skipping the height calculation.** Computing $h = b \sin A$ is not optional. It is the step that tells you how many triangles to solve for. Going straight to the Sine Law without this check means you may find and present an answer that does not actually exist.
- **Forgetting the second triangle.** When two triangles exist, you must solve both. Presenting only B_1 and stopping is an incomplete answer.
- **Not checking $A + B_2 < 180$ for Triangle 2.** Always verify that the second triangle's angles are valid before solving it. If $A + B_2 \geq 180$, discard it.
- **Applying the obtuse rule incorrectly.** When A is obtuse, the comparison is simply a vs. b — you do not need to compute h. Students sometimes compute h anyway and reach a wrong conclusion.
- **Mixing up a and b.** In SSA, the given angle A is always paired with side a (opposite), and b is the side adjacent to A. Swapping them changes the problem entirely.

Worksheet

For each problem: (i) compute h, (ii) state how many triangles exist and why, (iii) solve all valid triangles completely.

1. $A = 30$, $b = 10$, $a = 6$

2. $A = 30$, $b = 10$, $a = 14$

3. $A = 35$, $b = 12$, $a = 7$

4. $A = 50$, $b = 15$, $a = 20$

5. $A = 40$, $b = 14$, $a = 9$

6. $A = 120$, $b = 8$, $a = 5$

7. $A = 50$, $b = 13$, $a = 10$

8. **(Challenge)** $A = 38$, $b = 16$, $a = 10$. Solve all possible triangles. For each valid triangle, find all three angles and all three sides.

Answer Key

1. $A = 30$, $b = 10$, $a = 6$.

Step 1: Compute h.

$$h = 10 \times \sin 30 = 10 \times 0.5 = 5.000$$

Step 2: Compare. $h = 5 \leq a = 6 < b = 10 \quad \Rightarrow$ **two triangles.**
Step 3: Find $\sin B$.

$$\sin B = \frac{10 \times \sin 30}{6} = \frac{10 \times 0.5}{6} = 0.8333$$

Triangle 1:

$$B_1 = \sin^{-1}(0.8333) \approx 56.44, \quad C_1 = 180 - 30 - 56.44 = 93.56$$

$$c_1 = \frac{6 \times \sin 93.56}{\sin 30} = \frac{6 \times 0.9981}{0.5} \approx 11.98$$

Triangle 2:

$$B_2 = 180 - 56.44 = 123.56, \quad C_2 = 180 - 30 - 123.56 = 26.44$$

$$c_2 = \frac{6 \times \sin 26.44}{\sin 30} = \frac{6 \times 0.4452}{0.5} \approx 5.34$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 56.44,\ C_1 \approx 93.56,\ c_1 \approx 11.98 \\ \textbf{Triangle 2:} & B_2 \approx 123.56,\ C_2 \approx 26.44,\ c_2 \approx 5.34 \end{array}}$$

2. $A = 30$, $b = 10$, $a = 14$.

Step 1: Compute h.

$$h = 10 \times \sin 30 = 5.000$$

Step 2: Compare. $a = 14 \geq b = 10 \Rightarrow$ **one triangle.**
Step 3: Find B.

$$\sin B = \frac{10 \times \sin 30}{14} = \frac{5}{14} = 0.3571$$

$$B = \sin^{-1}(0.3571) \approx 20.92$$

Step 4: Find C and c.

$$C = 180 - 30 - 20.92 = 129.08$$

$$c = \frac{14 \times \sin 129.08}{\sin 30} = \frac{14 \times 0.7762}{0.5} \approx 21.74$$

$$\boxed{B \approx 20.92, \quad C \approx 129.08, \quad c \approx 21.74}$$

3. $A = 35$, $b = 12$, $a = 7$.

Step 1: Compute h.

$$h = 12 \times \sin 35 = 12 \times 0.5736 = 6.883$$

Step 2: Compare. $h = 6.883 \leq a = 7 < b = 12 \Rightarrow$ **two triangles.**
Step 3: Find $\sin B$**.**

$$\sin B = \frac{12 \times \sin 35}{7} = \frac{12 \times 0.5736}{7} = 0.9833$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9833) \approx 79.51, \quad C_1 = 180 - 35 - 79.51 = 65.49$$

$$c_1 = \frac{7 \times \sin 65.49}{\sin 35} = \frac{7 \times 0.9096}{0.5736} \approx 11.10$$

Triangle 2:

$$B_2 = 180 - 79.51 = 100.49, \quad C_2 = 180 - 35 - 100.49 = 44.51$$

$$c_2 = \frac{7 \times \sin 44.51}{\sin 35} = \frac{7 \times 0.7009}{0.5736} \approx 8.55$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 79.51,\ C_1 \approx 65.49,\ c_1 \approx 11.10 \\ \textbf{Triangle 2:} & B_2 \approx 100.49,\ C_2 \approx 44.51,\ c_2 \approx 8.55 \end{array}}$$

4. $A = 50$, $b = 15$, $a = 20$.

Step 1: Compute h.

$$h = 15 \times \sin 50 = 15 \times 0.7660 = 11.491$$

Step 2: Compare. $a = 20 \geq b = 15 \Rightarrow$ **one triangle.**
Step 3: Find B**.**

$$\sin B = \frac{15 \times \sin 50}{20} = \frac{15 \times 0.7660}{20} = 0.5745$$

$$B = \sin^{-1}(0.5745) \approx 35.07$$

Step 4: Find C and c.

$$C = 180 - 50 - 35.07 = 94.93$$

$$c = \frac{20 \times \sin 94.93}{\sin 50} = \frac{20 \times 0.9965}{0.7660} \approx 26.01$$

$$\boxed{B \approx 35.07, \quad C \approx 94.93, \quad c \approx 26.01}$$

5. $A = 40$, $b = 14$, $a = 9$.

Step 1: Compute h.

$$h = 14 \times \sin 40 = 14 \times 0.6428 = 8.999$$

Step 2: Compare. $h = 8.999 \leq a = 9 < b = 14 \Rightarrow$ **two triangles.**
Step 3: Find $\sin B$.

$$\sin B = \frac{14 \times \sin 40}{9} = \frac{14 \times 0.6428}{9} = 0.9999$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9999) \approx 89.16, \quad C_1 = 180 - 40 - 89.16 = 50.84$$

$$c_1 = \frac{9 \times \sin 50.84}{\sin 40} = \frac{9 \times 0.7762}{0.6428} \approx 10.86$$

Triangle 2:

$$B_2 = 180 - 89.16 = 90.84, \quad C_2 = 180 - 40 - 90.84 = 49.16$$

$$c_2 = \frac{9 \times \sin 49.16}{\sin 40} = \frac{9 \times 0.7559}{0.6428} \approx 10.59$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 89.16,\ C_1 \approx 50.84,\ c_1 \approx 10.86 \\ \textbf{Triangle 2:} & B_2 \approx 90.84,\ C_2 \approx 49.16,\ c_2 \approx 10.59 \end{array}}$$

6. $A = 120$, $b = 8$, $a = 5$.

Step 1: A is obtuse. Apply the obtuse rule.

$$a = 5 \leq b = 8$$

When A is obtuse, we need $a > b$ for a triangle to exist. Since $a \leq b$, no triangle is possible.

$$\boxed{\text{No triangle exists.}}$$

7. $A = 50,\ b = 13,\ a = 10.$

Step 1: Compute h.

$$h = 13 \times \sin 50 = 13 \times 0.7660 = 9.959$$

Step 2: Compare. $h = 9.959 \leq a = 10 < b = 13 \Rightarrow$ **two triangles.**
Step 3: Find $\sin B$.

$$\sin B = \frac{13 \times \sin 50}{10} = \frac{13 \times 0.7660}{10} = 0.9959$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9959) \approx 84.78, \quad C_1 = 180 - 50 - 84.78 = 45.22$$

$$c_1 = \frac{10 \times \sin 45.22}{\sin 50} = \frac{10 \times 0.7097}{0.7660} \approx 9.27$$

Triangle 2:

$$B_2 = 180 - 84.78 = 95.22, \quad C_2 = 180 - 50 - 95.22 = 34.78$$

$$c_2 = \frac{10 \times \sin 34.78}{\sin 50} = \frac{10 \times 0.5700}{0.7660} \approx 7.44$$

$$\boxed{\begin{array}{ll} \textbf{Triangle 1:} & B_1 \approx 84.78,\ C_1 \approx 45.22,\ c_1 \approx 9.27 \\ \textbf{Triangle 2:} & B_2 \approx 95.22,\ C_2 \approx 34.78,\ c_2 \approx 7.44 \end{array}}$$

8. (Challenge) $A = 38,\ b = 16,\ a = 10.$

Step 1: Compute h.

$$h = 16 \times \sin 38 = 16 \times 0.6157 = 9.851$$

Step 2: Compare. $h = 9.851 \leq a = 10 < b = 16 \Rightarrow$ **two triangles.**
Step 3: Find $\sin B$.

$$\sin B = \frac{16 \times \sin 38}{10} = \frac{16 \times 0.6157}{10} = 0.9851$$

Triangle 1:

$$B_1 = \sin^{-1}(0.9851) \approx 80.08, \quad C_1 = 180 - 38 - 80.08 = 61.92$$

$$c_1 = \frac{10 \times \sin 61.92}{\sin 38} = \frac{10 \times 0.8829}{0.6157} \approx 14.34$$

Triangle 2:

$$B_2 = 180 - 80.08 = 99.92, \quad C_2 = 180 - 38 - 99.92 = 42.08$$

$$c_2 = \frac{10 \times \sin 42.08}{\sin 38} = \frac{10 \times 0.6704}{0.6157} \approx 10.89$$

$$\boxed{\begin{array}{l} \textbf{Triangle 1:} \quad B_1 \approx 80.08,\ C_1 \approx 61.92,\ c_1 \approx 14.34 \\ \textbf{Triangle 2:} \quad B_2 \approx 99.92,\ C_2 \approx 42.08,\ c_2 \approx 10.89 \end{array}}$$

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