Triangles, Special Triangles, and Trigonometry Review
ApexMath

Notes

Why This Lesson Comes First

Every lesson in Unit 8 involves triangles. Before learning any new laws, it is essential to have the vocabulary, labelling system, and foundational trigonometry locked in. A single mislabelled side can make every step that follows incorrect. This lesson also introduces the exact trigonometric values from special triangles — values that appear constantly throughout the unit and that you are expected to recall without a calculator.

Triangle Vocabulary

- A triangle has three **vertices** (corners), three **sides**, and three **interior angles**.
- The angles in any triangle always add up to exactly 180:

$$A + B + C = 180$$

- An **acute triangle** has all three angles less than 90.
- An **obtuse triangle** has one angle greater than 90.
- A **right triangle** has one angle equal to exactly 90.

Standard Labelling Convention

In Unit 8, every triangle is labelled the same way. This convention is used in every lesson, every example, and every test question.

- Angles are labelled with capital letters: A, B, C.
- The side *opposite* angle A is labelled with the lowercase letter a.
- The side *opposite* angle B is labelled b.
- The side *opposite* angle C is labelled c.

This means side a and angle A always face each other across the triangle. This pairing is the foundation of the Sine Law in Lesson 2.

Right Triangle Trigonometry Review (SOHCAHTOA)

For a right triangle with an acute angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

To find a missing angle, use the inverse functions on your calculator:

$$\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right) \qquad \theta = \cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right) \qquad \theta = \tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right)$$

The 45-45-90 Special Triangle

A 45-45-90 triangle is an isosceles right triangle. Both legs are equal, and the hypotenuse is $\sqrt{2}$ times the length of each leg.

Side ratios: if each leg has length 1, then the hypotenuse has length $\sqrt{2}$.

$$\text{leg} : \text{leg} : \text{hypotenuse} = 1 : 1 : \sqrt{2}$$

Exact trigonometric values:

$$\sin 45 = \frac{\sqrt{2}}{2} \qquad \cos 45 = \frac{\sqrt{2}}{2} \qquad \tan 45 = 1$$

Scaling rule: multiply every side by the same number k. If the legs have length k, the hypotenuse is $k\sqrt{2}$. If the hypotenuse has length h, each leg is $\dfrac{h}{\sqrt{2}} = \dfrac{h\sqrt{2}}{2}$.

The 30-60-90 Special Triangle

A 30-60-90 triangle is formed by cutting an equilateral triangle exactly in half. The three sides are always in the ratio $1 : \sqrt{3} : 2$.

Side ratios: if the shortest side (opposite 30) has length 1, then:

$$\text{short} : \text{long} : \text{hypotenuse} = 1 : \sqrt{3} : 2$$

Exact trigonometric values:

$$\sin 30 = \frac{1}{2} \qquad \cos 30 = \frac{\sqrt{3}}{2} \qquad \tan 30 = \frac{\sqrt{3}}{3}$$

$$\sin 60 = \frac{\sqrt{3}}{2} \qquad \cos 60 = \frac{1}{2} \qquad \tan 60 = \sqrt{3}$$

Scaling rule: multiply every side by the same number k.

- Short leg (opp 30) $= k$
- Long leg (opp 60) $= k\sqrt{3}$
- Hypotenuse (opp 90) $= 2k$

Master Reference Table

Angle	**sin**	**cos**	**tan**
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Pro Tip

- To memorize the table, notice the pattern in the sine column: $\sin 30 = \frac{\sqrt{1}}{2}$, $\sin 45 = \frac{\sqrt{2}}{2}$, $\sin 60 = \frac{\sqrt{3}}{2}$. The numerator goes $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$ in order. The cosine column is the same pattern read backwards.
- In a 30-60-90 triangle, the shortest side is always opposite the smallest angle (30), and the longest leg is always opposite the largest acute angle (60). If you find yourself assigning a large side to the 30 angle, something is wrong.
- When a problem gives you the hypotenuse of a 45-45-90 triangle, divide by $\sqrt{2}$ to get the leg length. Always rationalize: $\dfrac{h}{\sqrt{2}} = \dfrac{h\sqrt{2}}{2}$.
- Always label a triangle diagram before doing any calculation. Write a, b, c on the sides and A, B, C at the angles. One minute of labelling prevents five minutes of errors.
- The angle sum property $A + B + C = 180$ is a free equation. Whenever you know two angles, the third is immediate. Use this to check your work at the end of every problem.

Examples

Example 1 (45-45-90 — Find Missing Sides): A right isosceles triangle has legs of length 7. Find the exact length of the hypotenuse.

Step 1: Identify the triangle type. Both acute angles are 45, so this is a 45-45-90 triangle.

$$\text{leg} : \text{leg} : \text{hyp} = 1 : 1 : \sqrt{2}$$

Step 2: The scaling factor is $k = 7$. Multiply the hypotenuse ratio by k.

$$\text{hyp} = k\sqrt{2} = 7\sqrt{2}$$

$$\boxed{\text{hyp} = 7\sqrt{2} \approx 9.90}$$

Example 2 (30-60-90 — Find Missing Sides from Hypotenuse): A 30-60-90 triangle has a hypotenuse of length 10. Find the lengths of both legs.

Step 1: The hypotenuse corresponds to the ratio value 2. Find the scaling factor k.

$$2k = 10 \qquad \Longrightarrow \qquad k = 5$$

Step 2: Apply the scaling factor to both legs.

$$\text{short leg (opp 30)} = k = 5$$

$$\text{long leg (opp 60)} = k\sqrt{3} = 5\sqrt{3}$$

Step 3: Verify with the Pythagorean theorem.

$$5^2 + (5\sqrt{3})^2 = 25 + 75 = 100 = 10^2 \checkmark$$

$$\boxed{\text{short leg} = 5, \quad \text{long leg} = 5\sqrt{3} \approx 8.66}$$

Example 3 (30-60-90 — Find Missing Sides from Short Leg): In a 30-60-90 triangle, the side opposite 30 has length 5. Find the hypotenuse and the side opposite 60.

Step 1: The short leg corresponds to ratio value 1, so $k = 5$.
Step 2: Apply the ratios.

$$\text{hyp} = 2k = 2 \times 5 = 10$$

$$\text{long leg} = k\sqrt{3} = 5\sqrt{3}$$

$$\boxed{\text{hyp} = 10, \quad \text{long leg} = 5\sqrt{3} \approx 8.66}$$

Example 4 (45-45-90 — Find Legs from Hypotenuse): A 45-45-90 triangle has a hypotenuse of length 12. Find the length of each leg.

Step 1: Use the scaling rule. The hypotenuse is $k\sqrt{2}$, so solve for k.

$$k\sqrt{2} = 12 \qquad \Longrightarrow \qquad k = \frac{12}{\sqrt{2}}$$

Step 2: Rationalize the denominator.

$$k = \frac{12}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

$$\boxed{\text{each leg} = 6\sqrt{2} \approx 8.49}$$

Example 5 (Using Exact Values in a Calculation): Without a calculator, find the exact value of:

$$\sin 60 \times \cos 30 + \tan 45$$

Step 1: Substitute the exact values from the reference table.

$$\sin 60 = \frac{\sqrt{3}}{2}, \qquad \cos 30 = \frac{\sqrt{3}}{2}, \qquad \tan 45 = 1$$

Step 2: Substitute and simplify.

$$\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + 1$$

$$= \frac{3}{4} + 1 = \frac{3}{4} + \frac{4}{4}$$

$$\boxed{\frac{7}{4}}$$

Common Mistakes

- **Confusing the short and long legs in a 30-60-90 triangle.** The short leg is always opposite 30 and the long leg is always opposite 60. Swapping them gives a completely wrong answer.
- **Forgetting to rationalize the denominator.** When a leg equals $\frac{h}{\sqrt{2}}$, always multiply top and bottom by $\sqrt{2}$ to write it as $\frac{h\sqrt{2}}{2}$. Leaving $\sqrt{2}$ in the denominator is considered an unsimplified answer.
- **Mislabelling sides and angles.** Side a must be opposite angle A. If you label the sides and angles in the wrong positions, every formula you apply will give the wrong answer.
- **Using a decimal approximation when an exact answer is required.** If a question asks for the exact value, your answer must contain $\sqrt{2}$ or $\sqrt{3}$ as needed — not a rounded decimal.
- **Applying SOHCAHTOA to a non-right triangle.** SOHCAHTOA only works when there is a 90 angle. For any other triangle, you need the Sine Law or Cosine Law (Lessons 2–5).

Worksheet

Part A: Special Triangle Side Lengths

1. A 30-60-90 triangle has a hypotenuse of length 8. Find the exact lengths of both legs.

2. A 45-45-90 triangle has legs of length 9. Find the exact length of the hypotenuse.

3. A 30-60-90 triangle has a long leg of length $6\sqrt{3}$. Find the exact lengths of the short leg and the hypotenuse.

4. A 45-45-90 triangle has a hypotenuse of length 10. Find the exact length of each leg.

5. A 30-60-90 triangle has a short leg of length 4. Find the exact lengths of the long leg and the hypotenuse.

6. A 45-45-90 triangle has a hypotenuse of length $6\sqrt{2}$. Find the exact length of each leg.

7. A 30-60-90 triangle has a long leg of length 10. Find the exact lengths of the short leg and the hypotenuse.
(Hint: long leg $= k\sqrt{3}$*, so solve for* k *first.)*

Part B: Exact Trigonometric Values

8. Fill in the exact values in the table below without using a calculator.

Angle	**sin**	**cos**	**tan**
30			
45			
60			

9. Without a calculator, evaluate the exact value of:

$$\sin 30 + \cos 60 + \tan 45$$

10. Without a calculator, evaluate the exact value of:

$$\cos 30 \times \sin 60 - \cos 45 \times \sin 45$$

Answer Key

1. 30-60-90 triangle, hypotenuse $= 8$.

Step 1: The hypotenuse corresponds to $2k$. Solve for k.

$$2k = 8 \qquad \Longrightarrow \qquad k = 4$$

Step 2: Apply the ratios.

$$\text{short leg (opp 30)} = k = 4$$

$$\text{long leg (opp 60)} = k\sqrt{3} = 4\sqrt{3}$$

$$\boxed{\text{short leg} = 4, \quad \text{long leg} = 4\sqrt{3} \approx 6.93}$$

2. 45-45-90 triangle, legs $= 9$.

Step 1: The scaling factor is $k = 9$. The hypotenuse is $k\sqrt{2}$.

$$\text{hyp} = 9\sqrt{2}$$

$$\boxed{\text{hyp} = 9\sqrt{2} \approx 12.73}$$

3. 30-60-90 triangle, long leg $= 6\sqrt{3}$.

Step 1: The long leg corresponds to $k\sqrt{3}$. Solve for k.

$$k\sqrt{3} = 6\sqrt{3} \qquad \Longrightarrow \qquad k = 6$$

Step 2: Apply the ratios.

$$\text{short leg} = k = 6$$

$$\text{hyp} = 2k = 12$$

$$\boxed{\text{short leg} = 6, \quad \text{hyp} = 12}$$

4. 45-45-90 triangle, hypotenuse $= 10$.

Step 1: The hypotenuse is $k\sqrt{2}$. Solve for k.

$$k\sqrt{2} = 10 \qquad \Longrightarrow \qquad k = \frac{10}{\sqrt{2}}$$

Step 2: Rationalize.

$$k = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

$$\boxed{\text{each leg} = 5\sqrt{2} \approx 7.07}$$

5. 30-60-90 triangle, short leg $= 4$.

Step 1: The short leg corresponds to k, so $k = 4$.
Step 2: Apply the ratios.

$$\text{long leg} = k\sqrt{3} = 4\sqrt{3}$$

$$\text{hyp} = 2k = 8$$

$$\boxed{\text{long leg} = 4\sqrt{3} \approx 6.93, \quad \text{hyp} = 8}$$

6. 45-45-90 triangle, hypotenuse $= 6\sqrt{2}$.

Step 1: Solve for k.

$$k\sqrt{2} = 6\sqrt{2} \qquad \Longrightarrow \qquad k = 6$$

Step 2: Each leg equals k.

$$\boxed{\text{each leg} = 6}$$

7. 30-60-90 triangle, long leg $= 10$.

Step 1: The long leg corresponds to $k\sqrt{3}$. Solve for k.

$$k\sqrt{3} = 10 \qquad \Longrightarrow \qquad k = \frac{10}{\sqrt{3}}$$

Step 2: Rationalize.

$$k = \frac{10}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

Step 3: Find the short leg and hypotenuse.

$$\text{short leg} = k = \frac{10\sqrt{3}}{3} \approx 5.77$$

$$\text{hyp} = 2k = \frac{20\sqrt{3}}{3} \approx 11.55$$

$$\boxed{\text{short leg} = \frac{10\sqrt{3}}{3}, \quad \text{hyp} = \frac{20\sqrt{3}}{3}}$$

8. Completed reference table.

Angle	**sin**	**cos**	**tan**
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

9. Evaluate $\sin 30 + \cos 60 + \tan 45$ exactly.

Step 1: Substitute exact values.

$$\sin 30 = \frac{1}{2}, \qquad \cos 60 = \frac{1}{2}, \qquad \tan 45 = 1$$

Step 2: Add.

$$\frac{1}{2} + \frac{1}{2} + 1 = 1 + 1$$

$$\boxed{2}$$

10. Evaluate $\cos 30 \times \sin 60 - \cos 45 \times \sin 45$ exactly.

Step 1: Substitute exact values.

$$\cos 30 = \frac{\sqrt{3}}{2}, \quad \sin 60 = \frac{\sqrt{3}}{2}, \quad \cos 45 = \frac{\sqrt{2}}{2}, \quad \sin 45 = \frac{\sqrt{2}}{2}$$

Step 2: Multiply each pair.

$$\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}$$

$$\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$$

Step 3: Subtract.

$$\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4}$$

$$\boxed{\frac{1}{4}}$$

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