Pre-Calculus 12 – Complete Study Notes
Lesson 5: Function Operations and Conics

Notes

Function Operations

(f + g)(x) = f(x) + g(x)
(f - g)(x) = f(x) - g(x)
(f $\cdot g)(x) = f(x)\, g(x)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$
(f $\circ g)(x) = f(g(x))$

Conic Sections Overview

Circle

$(x-h)^2 + (y-k)^2 = r^2$
Center = (h, k)
Radius = r

Ellipse (horizontal major axis)

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$
Center = (h, k)
Vertices = $(h \pm a,\ k)$
Co-vertices = $(h,\ k \pm b)$
Foci = $(h \pm c,\ k)$, where $c^2 = a^2 - b^2$
Eccentricity $e = \frac{c}{a}$

Ellipse (vertical major axis)

$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a > b$
Foci = $(h,\ k \pm c)$

Eccentricity $e = \dfrac{c}{a}$

Hyperbola (horizontal transverse axis)

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$
Foci $= (h \pm c,\, k)$, where $c^2 = a^2 + b^2$
Asymptotes: $y - k = \pm \dfrac{b}{a}(x-h)$
Eccentricity $e = \dfrac{c}{a}$

Hyperbola (vertical transverse axis)

$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
Foci $= (h,\, k \pm c)$
Asymptotes: $y - k = \pm \dfrac{a}{b}(x-h)$
Eccentricity $e = \dfrac{c}{a}$

Parabola (opening up/down)

$(x-h)^2 = 4p(y-k)$
Vertex $= (h, k)$
Focus $= (h,\, k+p)$
Directrix: $y = k - p$
Axis of symmetry: $x = h$

Parabola (opening left/right)

$(y-k)^2 = 4p(x-h)$
Focus $= (h+p,\, k)$
Directrix: $x = h - p$
Axis of symmetry: $y = k$

Walkthrough Example 1: Function Operations

Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = x - 4$. Find $(f \circ g)(x)$.

Step 1: Use the definition of composition: $(f \circ g)(x) = f(g(x))$.

Step 2: Substitute $g(x) = x - 4$ into f: $f(g(x)) = f(x-4)$.

Step 3: Replace x by $(x-4)$ in $f(x)$: $f(x-4) = 2(x-4)^2 - 3(x-4) + 1$.

Step 4: Expand $(x-4)^2$: $= 2(x^2 - 8x + 16) - 3x + 12 + 1$.

Step 5: Distribute and combine like terms: $= 2x^2 - 16x + 32 - 3x + 12 + 1$.

Step 6: Final simplify: $= 2x^2 - 19x + 45$.

Answer: $(f \circ g)(x) = 2x^2 - 19x + 45$.

Walkthrough Example 2: Conics — General Form to Standard Form

Convert $x^2 + y^2 - 6x + 8y + 9 = 0$ into standard form.

Step 1: Group x-terms and y-terms: $(x^2 - 6x) + (y^2 + 8y) + 9 = 0$.

Step 2: Complete the square for each group.
For $x^2 - 6x$: half of -6 is -3, square gives 9.
For $y^2 + 8y$: half of 8 is 4, square gives 16.

Step 3: Add inside, balance outside: $(x^2 - 6x + 9) + (y^2 + 8y + 16) + 9 - 9 - 16 = 0$.

Step 4: Rewrite as squares: $(x - 3)^2 + (y + 4)^2 - 16 = 0$.

Step 5: Move constant to the other side: $(x - 3)^2 + (y + 4)^2 = 16$.

Answer: Circle with center $(3, -4)$ and radius 4.

Worksheet

Function Operations

1. Let $f(x) = 3x + 2$ and $g(x) = x^2 - 1$. Find $(f \circ g)(x)$.
2. If $f(x) = \sqrt{x}$ and $g(x) = 2x + 3$, find $(g \circ f)(x)$.
3. Let $f(x) = 5x$ and $g(x) = x - 7$. Find $(f - g)(x)$.

Conics

4. Convert $x^2 + y^2 + 4x - 6y - 12 = 0$ into standard form. State the center and radius.
5. Write the equation of a parabola with vertex at $(2, -3)$ and focus at $(2, 1)$.
6. An ellipse has equation $\dfrac{(x-1)^2}{25} + \dfrac{(y+2)^2}{9} = 1$. Find the center, vertices, foci, and eccentricity.

Answer Key

1. $(f \circ g)(x) = f(x^2 - 1) = 3(x^2 - 1) + 2 = 3x^2 - 3 + 2 = 3x^2 - 1$.

2. $(g \circ f)(x) = g(\sqrt{x}) = 2\sqrt{x} + 3$.

3. $(f - g)(x) = 5x - (x - 7) = 4x + 7$.

4. $x^2 + y^2 + 4x - 6y - 12 = 0$.
$(x^2 + 4x + 4) + (y^2 - 6y + 9) - 4 - 9 - 12 = 0$.
$(x + 2)^2 + (y - 3)^2 - 25 = 0$.
$(x + 2)^2 + (y - 3)^2 = 25$.
Center $= (-2, 3)$, Radius $= 5$.

5. Vertex $(2, -3)$, Focus $(2, 1)$ so $p = 4$, opens upward.
Equation: $(x - 2)^2 = 4p(y + 3) = 16(y + 3)$.

6. $\dfrac{(x-1)^2}{25} + \dfrac{(y+2)^2}{9} = 1$.
Center $= (1, -2)$.
$a^2 = 25 \Rightarrow a = 5$.
$b^2 = 9 \Rightarrow b = 3$.
$c^2 = a^2 - b^2 = 25 - 9 = 16 \Rightarrow c = 4$.
Vertices: $(1 \pm 5, -2) \Rightarrow (-4, -2), (6, -2)$.
Foci: $(1 \pm 4, -2) \Rightarrow (-3, -2), (5, -2)$.
Eccentricity: $e = \dfrac{c}{a} = \dfrac{4}{5}$.