Pre-Calculus 12 – Complete Study Notes
Lesson 4: Exponential and Logarithmic Functions

Notes: Exponential and Logarithmic Functions

Exponential functions have the form:

$$y = a \cdot b^x$$

where a is the initial value and b is the growth ($b > 1$) or decay ($0 < b < 1$) factor.

Logarithmic functions are the inverses of exponential functions:

$$y = \log_b(x) \quad \text{is the inverse of} \quad y = b^x$$

Growth and Decay Equations

- Exponential Growth: $y = a(1 + r)^t$ a = initial amount, r = growth rate, t = time
- Exponential Decay: $y = a(1 - r)^t$ r = decay rate

Logarithmic Properties

- $\log_b(xy) = \log_b(x) + \log_b(y)$ - $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ - $\log_b(x^n) = n\log_b(x)$ - Change of base formula: $\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$

Walkthrough of Example Question

Example: A bacteria culture has 500 bacteria initially. The population doubles every 3 hours. Find an equation for the population and determine the population after 12 hours.

Step 1: Identify the variables and formula - Initial population $a = 500$ - Growth factor $b = 2$ (doubles) - Time t in hours - General formula: $P(t) = a \cdot b^{t/T}$, where T is the doubling time

Step 2: Plug in the doubling time - $T = 3$, so the equation is:

$$P(t) = 500 \cdot 2^{t/3}$$

Step 3: Determine population after 12 hours - Substitute $t = 12$:

$$P(12) = 500 \cdot 2^{12/3} = 500 \cdot 2^4 = 500 \cdot 16 = 8000$$

- Final answer: 8000 bacteria after 12 hours

Step 4: Optional – Using logarithms to solve for time - If we want to find when the population reaches 4000:

$$4000 = 500 \cdot 2^{t/3}$$

$$\frac{4000}{500} = 2^{t/3} \implies 8 = 2^{t/3}$$

$$2^3 = 2^{t/3} \implies t/3 = 3 \implies t = 9 \text{ hours}$$

Worksheet: Practice Questions

1. A car depreciates in value by 15% per year. If its initial value is $30,000, write an exponential decay equation for the car's value and find its value after 5 years.

2. The population of a town is 12,000 and grows at 4% per year. Derive the exponential growth equation and determine the population after 10 years.

3. A radioactive substance decays at a rate of 5% per hour. If the initial mass is 200 grams, find the mass remaining after 8 hours.

4. A culture of bacteria triples every 6 hours. If the initial population is 800, write the exponential growth equation and determine the population after 18 hours.

5. A bank account has $5,000 initially and earns 3% interest compounded yearly. Write an equation for the account balance and find the balance after 7 years.

Answer Key

Q1: - Initial value $a = 30,000$, decay rate $r = 0.15$ - Exponential decay equation: $V(t) = 30000 \cdot (1 - 0.15)^t = 30000 \cdot 0.85^t$ - Value after 5 years:

$$V(5) = 30000 \cdot 0.85^5 = 30000 \cdot 0.4437 \approx 13311$$

- Car is worth approximately $13,311

Q2: - Initial population $a = 12000$, growth rate $r = 0.04$ - Exponential growth equation: $P(t) = 12000 \cdot 1.04^t$ - Population after 10 years:

$$P(10) = 12000 \cdot 1.04^{10} \approx 12000 \cdot 1.4802 \approx 17762$$

- Population 17,762

Q3: - Initial mass $a = 200$, decay rate $r = 0.05$ - Exponential decay equation: $M(t) = 200 \cdot 0.95^t$ - Mass after 8 hours:

$$M(8) = 200 \cdot 0.95^8 \approx 200 \cdot 0.6634 \approx 132.7$$

- Remaining mass 132.7 grams

Q4: - Initial population $a = 800$, triples every 6 hours, growth factor $b = 3$ - Equation: $P(t) = 800 \cdot 3^{t/6}$ - After 18 hours: $t/6 = 18/6 = 3$

$$P(18) = 800 \cdot 3^3 = 800 \cdot 27 = 21600$$

- Population = 21,600

Q5: - Initial balance $a = 5000$, interest rate $r = 0.03$ - Equation: $A(t) = 5000 \cdot 1.03^t$ - Balance after 7 years:

$$A(7) = 5000 \cdot 1.03^7 \approx 5000 \cdot 1.22987 \approx 6149$$

- Account balance $6,149