Pre-Calculus 12 Comprehensive Review

ApexMath

Unit 1: Arithmetic and Geometric Sequences and Series

Notes

- Arithmetic Sequence: $a_n = a_1 + (n-1)d$
- Arithmetic Series: $S_n = \frac{n}{2}(a_1 + a_n)$
- Geometric Sequence: $a_n = a_1 r^{n-1}$
- Geometric Series: $S_n = a_1 \frac{1-r^n}{1-r}, r \neq 1$

Example: Bouncing Ball

A ball is dropped from a height of 10 m. Each bounce reaches $\frac{3}{4}$ of the previous height. Find the total distance after 5 bounces.

First drop: 10 m

Bounces: geometric series with $a_1 = 10 * (3/4) = 7.5, r = 3/4, n = 5$

$$S_5 = 7.5\frac{1-(3/4)^5}{1-3/4} \approx 35.7$$

Total distance $= 10 + 35.7 \approx 45.7$ m

Total distance ≈ 45.7 m

Unit 2: Transformations

Notes

- Horizontal shift: $f(x-h)$ moves right by h
- Vertical shift: $f(x)+k$ moves up by k

- Reflection: $-f(x)$ reflects over x-axis, $f(-x)$ over y-axis
- Vertical stretch/compression: $af(x)$ stretches if $|a| > 1$, compresses if $0 < |a| < 1$

Example

$f(x) = x^2$ transformed to $-2(x-3)^2 + 4$, find $f(-1)$:

$$f(-1) = -2((-1-3)^2) + 4 = -2(16) + 4 = -32 + 4 = -28$$

$$\boxed{f(-1) = -28}$$

Unit 3: Polynomials

Notes

- Factor theorem: if $f(c) = 0$, $(x - c)$ is a factor
- End behavior depends on leading term
- Synthetic division for $f(x) \div (x - c)$

Example: Synthetic Division

Divide $f(x) = 2x^3 - 3x^2 + 4x - 5$ by $x - 2$:

2	2	-3	4	-5
		4	2	12
	2	1	6	7

Quotient $= 2x^2 + x + 6$, remainder $= 7$

$$\boxed{f(x) = (x-2)(2x^2 + x + 6) + 7}$$

Unit 4: Exponential and Logarithmic Functions

Notes

- Exponential: $f(x) = a \cdot b^x$
- Logarithm: $\ln(x)$ natural, $\log_b(x)$ base b
- Properties: $\log_b(xy) = \log_b x + \log_b y$, $\log_b(x^n) = n \log_b x$

Example: Population Growth

Population 1000 grows 5% annually, find t for 2000:

$$2000 = 1000(1.05)^t \implies t = \frac{\ln 2}{\ln 1.05} \approx 14.21$$

$\boxed{t \approx 14.2 \text{ years}}$

Unit 5: Function Operations and Conics

Notes

- $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $(f/g)(x)$
- Circle: $(x-h)^2 + (y-k)^2 = r^2$, Parabola: $y = a(x-h)^2 + k$

Example

Parabola: $y = 0.5(x-2)^2 + 1$, find y when $x = 5$:

$$y = 0.5(5-2)^2 + 1 = 0.5 * 9 + 1 = 5.5$$

$\boxed{y = 5.5}$

Unit 6: Trigonometric Functions

Notes

- $f(x) = a\sin(bx+c) + d$ or $f(x) = a\cos(bx+c) + d$
- Amplitude a = (max-min)/2, midline d = (max+min)/2
- Period $2\pi/b$, phase shift $-c/b$

Example: Ferris Wheel

Radius 25 m, center 27 m, period 40 s, start at lowest point:

$$b = 2\pi/40 = \pi/20, \quad h(t) = -25\cos(\pi t/20) + 27$$

$$h(15) = -25\cos(3\pi/4) + 27 = -25(-\sqrt{2}/2) + 27 \approx 44.7$$

$\boxed{h(15) \approx 44.7 \text{ m}}$

Unit 7: Trigonometric Equations and Identities

Notes

- Solve $\sin x = k$, $\cos x = k$, $\tan x = k$
- ASTC to determine quadrant: A=All, S=Sine, T=Tangent, C=Cosine positive
- Identities: Pythagorean, reciprocal, co-function, sum/difference, double angle
- arcsin, arccos, arctan are inverse trig functions
- $\pm$ means both positive and negative solutions in the specified quadrant

Example: Solve Equation

$\sin 2x = \sqrt{3}/2 \implies 2x = \pi/3, 2\pi/3 \implies x = \pi/6, \pi/3$

$\boxed{x = \pi/6, \pi/3}$

Worksheet (Extended)

1. Bouncing ball: dropped 12 m, 80% each bounce, distance after 6 bounces.
2. Arithmetic series: sum first 50 terms of $2, 5, 8, ...$
3. Geometric series: sum first 8 terms of $3, 6, 12, ...$
4. Find 10th term of arithmetic sequence 5, 8, 11,...
5. Find 7th term of geometric sequence 2, 6, 18,...
6. Transform $f(x) = x^2$: reflect over x-axis, vertical stretch 2, shift right 3, up 1. Find $f(0)$
7. Sketch $y = \frac{1}{2}(-x + 2)^2 + 3$
8. Synthetic division: $2x^3 + 3x^2 - 5x + 6$ by $x - 1$
9. Factor $x^3 - 4x^2 - x + 4$ using factor theorem
10. Find end behavior of $f(x) = -3x^4 + 2x^3 - x + 5$
11. Population grows 1000 at 6% per year. Find t for 2500.
12. Solve $\ln(2x - 1) = 3$
13. Solve $3^x = 81$
14. $(f/g)(2)$, $f(x) = x^2 + 1$, $g(x) = 2x - 3$

15. Parabola: $y = 2(x-1)^2 + 3$, find y when $x = 4$
16. Circle: $(x-3)^2 + (y+2)^2 = 16$, find y when $x = 5$
17. Ferris wheel: radius 15 m, center 17 m, period 30 s, start at highest point. Find $h(t)$, $h(12)$
18. Tidal buoy: high 9 m, low 3 m, period 12 h, start high. Find $B(t)$, $B(7)$
19. Sunlight: max 16 h, min 8 h, period 365 days, min at day 355. Find $D(t)$ and $D(90)$
20. Solve $\sin x = 0.707$
21. Solve $\cos 2x = 0.5$
22. Verify $1 + \tan^2(\pi/6) = \sec^2(\pi/6)$
23. Solve $\sin(x + \pi/4) = \sqrt{2}/2$
24. Solve $\tan^2 x - 3 = 0$

Answer Key (Very Detailed Step-by-Step)

1. Bouncing ball: $a_1 = 12 * 0.8 = 9.6$, $r = 0.8$, $n = 6$ $S_6 = 9.6(1 - 0.8^6)/(1 - 0.8) \approx 42.4$ m, add first drop 12 $\implies$ 54.4 m

   $\boxed{54.4 \text{ m}}$

2. $S_{50} = 50/2 * (2 + (2 + 49 * 3)) = 25 * 149 = 3725$

   $\boxed{3725}$

3. $S_8 = 3(1 - 2^8)/(1 - 2) = 765$

   $\boxed{765}$

4. $a_{10} = 5 + (10 - 1) * 3 = 5 + 27 = 32$

   $\boxed{32}$

5. $a_7 = 2 * 6^6 = 2 * 6^6 = 46656$

   $\boxed{46656}$

6. $f(x) = 2(-x + 3)^2 + 1$, $f(0) = 2(-0 + 3)^2 + 1 = 19$

   $\boxed{19}$

7. Sketch answer: parabola opens downwards, vertex at (2,3), compressed vertically 1/2

8. Synthetic: Quotient $2x^2 + 5x$, remainder 6

   $\boxed{f(x) = (x-1)(2x^2+5x) + 6}$

9. Factor: $(x-1)(x+1)(x-4)$

   $\boxed{(x-1)(x+1)(x-4)}$

10. End behavior: leading term $-3x^4$, $x \to \infty, f(x) \to -\infty$, $x \to -\infty, f(x) \to -\infty$

11. $t = \ln 2.5 / \ln 1.06 \approx 15.2$ yrs

    $\boxed{15.2 \text{ yrs}}$

12. $\ln(2x-1) = 3 \implies 2x - 1 = e^3 \implies x = (1+e^3)/2 \approx 10.542$

    $\boxed{10.542}$

13. $3^x = 81 \implies 3^x = 3^4 \implies x = 4$

    $\boxed{4}$

14. $(f/g)(2) = (2^2+1)/(2*2-3) = 5/1 = 5$

    $\boxed{5}$

15. $y = 2(4-1)^2 + 3 = 2*9 + 3 = 21$

    $\boxed{21}$

16. Circle: $(5-3)^2 + (y+2)^2 = 16 \implies 4 + (y+2)^2 = 16 \implies (y+2)^2 = 12 \implies y = -2 \pm 2\sqrt{3}$

    $\boxed{y = -2 \pm 2\sqrt{3}}$

17. Ferris: $b = 2\pi/30 = \pi/15$, $h(t) = 15\cos(\pi t/15) + 17$, $h(12) = 15\cos(4\pi/5) + 17 \approx 4.86$

    $\boxed{4.86 \text{ m}}$

18. Buoy: $a = 3, d = 6, b = \pi/6$, $B(t) = 3\cos(\pi t/6) + 6$, $B(7) = 3\cos(7\pi/6) + 6 \approx 3.40$

    $\boxed{3.40 \text{ m}}$

19. Sunlight: $a = (16-8)/2 = 4, d = (16+8)/2 = 12$, $b = 2\pi/365$, $D(t) = 12 - 4\cos(2\pi/365(t-355))$, $D(90) \approx 12.64$ h

    $\boxed{12.64 \text{ h}}$

20. $\sin x = 0.707 \implies x = \pi/4, 3\pi/4 + 2\pi n$

$\boxed{x = \pi/4, 3\pi/4 + 2\pi n}$

21. $\cos 2x = 0.5 \implies 2x = \pi/3, 5\pi/3 \implies x = \pi/6, 5\pi/6 + \pi n$

$\boxed{x = \pi/6, 5\pi/6 + \pi n}$

22. $1 + \tan^2(\pi/6) = 1 + 1/3 = 4/3 = \sec^2(\pi/6)$

$\boxed{4/3}$

23. $\sin(x + \pi/4) = \sqrt{2}/2 \implies x + \pi/4 = \pi/4, 3\pi/4 \implies x = 0, \pi/2 + 2\pi n$

$\boxed{x = 0, \pi/2 + 2\pi n}$

24. $\tan^2 x - 3 = 0 \implies \tan x = \pm\sqrt{3} \implies x = \pi/3, 2\pi/3 + \pi n$

$\boxed{x = \pi/3, 2\pi/3 + \pi n}$