Lesson 7: Trigonometric Equations and Identities

ApexMath

Notes

1) Trigonometric Equations

Basic forms: $\sin x = k$, $\cos x = k$, $\tan x = k$.

General solutions:

$$\begin{aligned}
\sin x = k &\implies x = \arcsin k + 2\pi n \quad \text{or} \quad x = \pi - \arcsin k + 2\pi n \\
\cos x = k &\implies x = \arccos k + 2\pi n \quad \text{or} \quad x = -\arccos k + 2\pi n \\
\tan x = k &\implies x = \arctan k + \pi n
\end{aligned}$$

$n \in \mathbb{Z}$.

The upside-down plus-minus sign $\mp$ means the opposite of the corresponding $\pm$ sign. Example: $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

Use ASTC to determine quadrants: A = All positive, S = Sine positive, T = Tangent positive, C = Cosine positive. Example: if $\sin x > 0$ and $\cos x < 0$, angle is in quadrant II.

Note: arcsin, arccos, and arctan are the same as $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

2) Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

3) Reciprocal Identities

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

4) Co-Function Identities

$$\begin{aligned}
\sin\left(\frac{\pi}{2} - x\right) &= \cos x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x \\
\tan\left(\frac{\pi}{2} - x\right) &= \cot x, \quad \cot\left(\frac{\pi}{2} - x\right) = \tan x
\end{aligned}$$

5) Double Angle Identities

$$\sin 2x = 2\sin x\cos x, \quad \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

6) Sum and Difference Identities

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y, \quad \cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

Walkthrough of Example Questions

Example 1: Solve a basic trig equation

$\sin x = 0.5$:

$$x = \arcsin 0.5 = \pi/6, \quad x = \pi - \pi/6 = 5\pi/6$$

$\boxed{x = \pi/6 + 2\pi n \text{ or } x = 5\pi/6 + 2\pi n}$

Example 2: Solve using identity

$\cos^2 x - \cos x - 2 = 0$:

$$u = \cos x \implies u^2 - u - 2 = 0 \implies (u-2)(u+1) = 0$$

$$\cos x = -1 \implies x = \pi + 2\pi n$$

$\boxed{x = \pi + 2\pi n}$

Example 3: Solve using double angle

$\sin 2x = \sqrt{3}/2$:

$$2x = \pi/3, 2\pi/3, 7\pi/3, 5\pi/3 \implies x = \pi/6 + \pi n, x = \pi/3 + \pi n$$

$\boxed{x = \pi/6 + \pi n \text{ or } x = \pi/3 + \pi n}$

Worksheet

A & B: Solve the equations and apply identities

1. $\sin x = 0.707$
2. $\cos x = -0.5$
3. $\tan x = 1$

4. $2\sin^2 x - 3\sin x + 1 = 0$
5. $\cos 2x = 0.5$
6. Solve $\sin(x + \pi/4) = \sqrt{2}/2$
7. Solve $\cos(x - \pi/6) = -\sqrt{3}/2$
8. Use the double-angle identity: $\sin 2x$ when $\sin x = 3/5$ and $\cos x > 0$
9. Find exact value of $\sin(\pi/3 + \pi/4)$ using sum identity
10. Verify $1 + \tan^2 x = \sec^2 x$ for $x = \pi/6$

C: Real-World Application Problems

1. Ferris wheel: radius 20 m, center 25 m, period 50 s, rider at lowest point. Derive $h(t)$ and find t when $h(t) = 40$.
2. Daylight hours: max 15, min 9, period 365 days, minimum day 355. Derive $D(t)$ and find t for $D(t) = 12$.
3. Buoy: high 8 m, low 2 m, period 12 h, start at high tide. Derive $B(t)$ and find t for $B(t) = 5$.

Answer Key: Step-by-Step Solutions

A & B Solutions

1. $\sin x = 0.707$
   $x = \arcsin 0.707 \approx \pi/4$ (Quadrant I), also $\pi - \pi/4 = 3\pi/4$ (Quadrant II)

   $$\boxed{x \approx \pi/4 + 2\pi n \text{ or } 3\pi/4 + 2\pi n}$$

2. $\cos x = -0.5$
   $x = \arccos(-0.5) = 2\pi/3$ (Quadrant II), also $x = -2\pi/3 + 2\pi = 4\pi/3$ (Quadrant III)

   $$\boxed{x = 2\pi/3 + 2\pi n \text{ or } 4\pi/3 + 2\pi n}$$

3. $\tan x = 1$
   $x = \arctan 1 = \pi/4$ (Quadrant I), also $\pi + \pi/4 = 5\pi/4$ (Quadrant III)

   $$\boxed{x = \pi/4 + \pi n \text{ or } 5\pi/4 + \pi n}$$

4. $2\sin^2 x - 3\sin x + 1 = 0$
Let $u = \sin x$, then $2u^2 - 3u + 1 = 0 \implies (2u-1)(u-1) = 0$
$u = 1/2 \implies x = \pi/6, 5\pi/6$; $u = 1 \implies x = \pi/2$

$$\boxed{x = \pi/6 + 2\pi n, 5\pi/6 + 2\pi n, \pi/2 + 2\pi n}$$

5. $\cos 2x = 0.5$
$2x = \arccos 0.5 = \pi/3$ (Quadrant I), $2x = 2\pi - \pi/3 = 5\pi/3$ (Quadrant IV)
$x = \pi/6 + \pi n, 5\pi/6 + \pi n$

$$\boxed{x = \pi/6 + \pi n \text{ or } 5\pi/6 + \pi n}$$

6. $\sin(x + \pi/4) = \sqrt{2}/2$
$x + \pi/4 = \pi/4 \implies x = 0$, also $x + \pi/4 = 3\pi/4 \implies x = \pi/2$

$$\boxed{x = 0 + 2\pi n \text{ or } \pi/2 + 2\pi n}$$

7. $\cos(x - \pi/6) = -\sqrt{3}/2$
$x - \pi/6 = 5\pi/6 \implies x = 5\pi/6 + \pi/6 = \pi$ $x - \pi/6 = 7\pi/6 \implies x = 7\pi/6 + \pi/6 = 4\pi/3$

$$\boxed{x = \pi + 2\pi n \text{ or } 4\pi/3 + 2\pi n}$$

8. $\sin 2x = 2\sin x \cos x$, $\sin x = 3/5$, $\cos x > 0$
$\sin 2x = 2 * (3/5) * \sqrt{1 - (3/5)^2} = 2 * (3/5) * (4/5) = 24/25$

$$\boxed{\sin 2x = 24/25}$$

9. $\sin(\pi/3 + \pi/4) = \sin \pi/3 \cos \pi/4 + \cos \pi/3 \sin \pi/4 = (\sqrt{3}/2 * \sqrt{2}/2) + (1/2 * \sqrt{2}/2) = (\sqrt{6} + \sqrt{2})/4$

$$\boxed{\sin(\pi/3 + \pi/4) = (\sqrt{6} + \sqrt{2})/4}$$

10. Verify $1 + \tan^2(\pi/6) = \sec^2(\pi/6)$
$\tan(\pi/6) = 1/\sqrt{3} \implies 1 + (1/3) = 4/3$, $\sec(\pi/6) = 2/\sqrt{3} \implies \sec^2(\pi/6) = 4/3$

$$\boxed{1 + \tan^2(\pi/6) = \sec^2(\pi/6)}$$

C: Real-World Applications

Example 1: Ferris Wheel

1. Amplitude: $a = 20$, Midline: $d = 25$
2. Period: 50 s $\implies b = 2\pi/50 = \pi/25$
3. Rider starts at lowest point $\implies h(t) = -20\cos(\pi t/25) + 25$
4. Solve $h(t) = 40$: $40 = -20\cos(\pi t/25) + 25 \implies \cos(\pi t/25) = -3/4$
5. Quadrants II and III: $\theta = \arccos(-3/4) \approx 2.418$, $2\pi - 2.418 \approx 3.865$
6. Solve for t: $t = 25 * 2.418/\pi \approx 19.2$ s, $t = 25 * 3.865/\pi \approx 30.7$ s
7. $\boxed{t \approx 19.2 \text{ s or } 30.7 \text{ s}}$

Example 2: Daylight Hours

1. Amplitude: $a = (15-9)/2 = 3$, Midline: $d = (15+9)/2 = 12$
2. Frequency factor: $b = 2\pi/365$
3. Minimum at $t = 355$: $D(t) = 12 - 3\cos(2\pi/365 * (t - 355))$
4. Solve $D(t) = 12$: $\cos(2\pi/365 * (t - 355)) = 0$
5. $\theta = \pi/2$ (Quadrant I) and $3\pi/2$ (Quadrant III)
6. Solve for t: $t - 355 = 365/4 \implies t \approx 81$, $t - 355 = 3 * 365/4 \implies t \approx 263$
7. $\boxed{t \approx 81 \text{ or } 263}$

Example 3: Buoy and Tide

1. Amplitude: $a = (8-2)/2 = 3$, Midline: $d = (8+2)/2 = 5$
2. Frequency factor: $b = 2\pi/12 = \pi/6$
3. Cosine function (starts at high): $B(t) = 3\cos(\pi t/6) + 5$
4. Solve $B(t) = 5$: $3\cos(\pi t/6) + 5 = 5 \implies \cos(\pi t/6) = 0$
5. Quadrants II and III: $\theta = \pi/2, 3\pi/2$
6. Solve for t: t=3 h, t=9 h
7. $\boxed{t = 3 \text{ h or } 9 \text{ h}}$