Lesson 6: Trigonometric Functions — Modeling Cyclical Phenomena

Notes

1) What is a trigonometric function?

A trigonometric function models periodic or cyclical behavior. Common examples include:

$$f(x) = a\sin(bx + c) + d$$

$$g(x) = a\cos(bx + c) + d$$

where:

$$\begin{aligned} a &= \text{amplitude} = \frac{\text{max} - \text{min}}{2} \\ b &= \text{frequency factor (number of cycles in } 2\pi) \\ \text{Period} &= \frac{2\pi}{b} \\ c &= \text{phase shift (horizontal)} \\ d &= \text{vertical displacement (midline)} = \frac{\text{max} + \text{min}}{2} \end{aligned}$$

2) Key Features

- Amplitude: maximum distance from midline
- Period: length of one full cycle
- Midline: horizontal axis around which the graph oscillates
- Phase shift: starting point of the cycle

3) Modeling Real-World Cyclical Problems

Steps:

1. Find maximum and minimum → amplitude A and midline d.
2. Determine full cycle → period P, then $b = 2\pi/P$.

3. Determine starting point and direction $\to$ choose sine or cosine, find phase shift c.
4. Write the equation.
5. Solve for required input values.

Worksheet: Modeling and Solving

Part 1: Model and Solve

1. A Ferris wheel has a radius of 15 m and its center is 17 m above the ground. It completes a rotation every 30 seconds. A rider starts at the highest point when $t = 0$.
   (a) Derive the trigonometric function $h(t)$ for the rider's height.
   (b) Find the rider's height after 12 seconds.
2. A city's daylight varies from 14 hours (max) to 10 hours (min). The shortest day is on day 355. Write a cosine model $D(t)$ with $t = 0$ on January 1. Find the daylight hours on day 180.
3. A buoy in the ocean moves with the tide. At high tide it is 9 m above a reference, at low tide 3 m. The tidal cycle is 12 hours, and at $t = 0$, the buoy is at maximum height.
   (a) Derive $B(t)$.
   (b) Find the height at $t = 7$ hours.
4. A Ferris wheel has a diameter of 40 m. The bottom is 2 m above the ground, and it makes one rotation every 60 seconds. A rider starts at the lowest point.
   (a) Write the equation for height $H(t)$.
   (b) Determine when the rider will be 30 m above the ground for the first time.
5. A spotlight oscillates in a sine wave pattern. It points at a wall such that the angle $\theta(t)$ varies between $20°$ and $80°$ every 10 seconds. At $t = 0$, it points at its maximum angle.
   (a) Write the trigonometric model for $\theta(t)$.
   (b) Solve for t when the angle is $50°$ for the first time.

Answer Key: Step-by-Step Solutions

1) Ferris Wheel (Radius 15 m, Center 17 m, Period 30 s, starts at max)

$$
\begin{aligned}
A &= 15 \\
d &= 17 \\
b &= \frac{2\pi}{30} = \frac{\pi}{15} \\
\text{Equation: } h(t) &= 15\cos\left(\frac{\pi}{15}t\right) + 17 \\
\text{Height at } t = 12: h(12) &= 15\cos\left(\frac{\pi}{15}\cdot 12\right) + 17 \\
&= 15\cos\left(\frac{12\pi}{15}\right) + 17 \\
&= 15\cos\left(\frac{4\pi}{5}\right) + 17 \\
\cos\left(\frac{4\pi}{5}\right) &\approx -0.809 \\
h(12) &\approx 15(-0.809) + 17 = -12.14 + 17 \approx 4.86 \text{ m}
\end{aligned}
$$

2) Daylight Model (Max 14, Min 10, shortest day 355)

$$
\begin{aligned}
A &= \frac{14-10}{2} = 2 \\
d &= \frac{14+10}{2} = 12 \\
b &= \frac{2\pi}{365} \\
D(t) &= 12 - 2\cos\left(\frac{2\pi}{365}(t-355)\right) \\
D(180) &= 12 - 2\cos\left(\frac{2\pi}{365}(180-355)\right) \\
&= 12 - 2\cos\left(\frac{-350\pi}{365}\right) \\
&\approx 12 - 2\cdot(-0.99) = 12 + 1.98 \approx 13.98 \text{ hours}
\end{aligned}
$$

3) Buoy (High 9, Low 3, Period 12)

$$A = \frac{9-3}{2} = 3$$
$$d = \frac{9+3}{2} = 6$$
$$b = \frac{2\pi}{12} = \frac{\pi}{6}$$
$$B(t) = 3\cos\left(\frac{\pi}{6}t\right) + 6$$
$$B(7) = 3\cos\left(\frac{\pi}{6}\cdot 7\right) + 6$$
$$= 3\cos\left(\frac{7\pi}{6}\right) + 6$$
$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \approx -0.866$$
$$B(7) \approx 3(-0.866) + 6 = -2.598 + 6 \approx 3.40 \text{ m}$$

4) Ferris Wheel (Diameter 40 m, Bottom 2 m, Period 60 s, starts at min)

$$A = \frac{40}{2} = 20$$
$$d = 2 + 20 = 22$$
$$b = \frac{2\pi}{60} = \frac{\pi}{30}$$
$$H(t) = -20\cos\left(\frac{\pi}{30}t\right) + 22 \quad \text{(starts at min)}$$
$$\text{Solve for } H(t) = 30: -20\cos\left(\frac{\pi}{30}t\right) + 22 = 30$$
$$-20\cos\left(\frac{\pi}{30}t\right) = 8$$
$$\cos\left(\frac{\pi}{30}t\right) = -0.4$$
$$\frac{\pi}{30}t = \arccos(-0.4)$$
$$t \approx 30\cdot\frac{1.9823}{\pi} \approx 18.9 \text{ s}$$

5) Spotlight Oscillation (Max 80°, Min 20°, Period 10 s, starts at max)

$$
\begin{aligned}
A &= \frac{80-20}{2} = 30 \\
d &= \frac{80+20}{2} = 50 \\
b &= \frac{2\pi}{10} = \frac{\pi}{5} \\
\theta(t) &= 50 + 30\cos\left(\frac{\pi}{5}t\right) \\
\text{Solve for } \theta = 50 : 50 + 30\cos\left(\frac{\pi}{5}t\right) &= 50 \\
30\cos\left(\frac{\pi}{5}t\right) &= 0 \\
\cos\left(\frac{\pi}{5}t\right) &= 0 \\
\frac{\pi}{5}t &= \frac{\pi}{2} \\
t &= 2.5 \text{ s (first occurrence)}
\end{aligned}
$$