

Pre-Calculus 12

Comprehensive Review Notes

Walkthrough Questions Included

1. Functions & Relations

Key Concepts:

- **Domain:** Exclude values making denominator zero or negative radicands.
- **Vertical asymptotes:** Set denominator equal to zero (if factor does not cancel).
- **Holes:** Occur when factors cancel in numerator and denominator.
- **Horizontal asymptote:** Compare degrees of numerator and denominator.
 - If degrees equal, horizontal asymptote is ratio of leading coefficients.
 - If numerator degree less, asymptote at $y = 0$.
 - If numerator degree greater, no horizontal asymptote (maybe slant).

Walkthrough Question 1:

Given:

$$f(x) = \frac{x^2 - 9}{x^2 - 4x + 3}$$

- Find the domain of f .
- Identify vertical asymptotes and holes.
- Find the horizontal asymptote.
- Describe behavior near the vertical asymptotes.

Solution:

- Factor numerator and denominator:

$$x^2 - 9 = (x - 3)(x + 3), \quad x^2 - 4x + 3 = (x - 3)(x - 1).$$

- Simplify:

$$f(x) = \frac{x+3}{x-1}, \quad x \neq 3.$$

- Domain excludes $x = 1$ (vertical asymptote) and $x = 3$ (hole).
- Vertical asymptote at $x = 1$.
- Hole at $x = 3$, find $f(3) = \frac{3+3}{3-1} = 3$.
- Horizontal asymptote: degrees equal (both 1 after simplification), ratio leading coefficients = 1, so $y = 1$.
- Behavior near $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty.$$

2. Polynomial & Rational Functions

Key Concepts:

- Use synthetic division to test factors quickly.
- If remainder = 0, divisor is a factor.
- Numerical root finding is practical for high-degree polynomials.

Walkthrough Question 2:

Given:

$$p(x) = 2x^4 - 3x^3 - 11x^2 + 6x + 18,$$

- Use synthetic division to test if $x - 2$ is a factor.
- Find approximate zeros of $p(x)$.

Solution:

- Coefficients: 2, -3, -11, 6, 18.
- Synthetic division by $x - 2$:

$$\begin{array}{r|rrrrr} 2 & 2 & -3 & -11 & 6 & 18 \\ & & 4 & 2 & -18 & -24 \\ \hline & 2 & 1 & -9 & -12 & -6 \end{array}$$

- Remainder = $-6 \neq 0$, so $x - 2$ is not a factor.

- Numerical roots (using calculator/software):

$$x \approx 2.666, \quad 1.613, \quad -1.390 \pm 0.402i.$$

3. Exponents & Logarithms

Key Concepts:

- Use logarithm properties to solve exponential equations.
- Apply change of base for logarithms.
- Combine logarithms when possible.

Walkthrough Question 3:

Solve for x :

$$5^{2x-1} = 7.$$

Solution:

$$\begin{aligned} 2x - 1 &= \log_5 7 \\ x &= \frac{1 + \log_5 7}{2}. \end{aligned}$$

Using change of base:

$$\log_5 7 = \frac{\ln 7}{\ln 5} \approx 1.209,$$

so

$$x \approx \frac{1 + 1.209}{2} = 1.1045.$$

4. Sequences & Series

Key Concepts:

- Geometric sequences multiply by a common ratio r .
- Total distance traveled by bouncing balls can be modeled as geometric series.

Walkthrough Question 4:

A ball is dropped from 5 m height and bounces back to 80% of the previous height each time.

- (a) What is the total distance traveled by the ball by the time it hits the ground after the 5th bounce?

Solution:

- First drop: 5 m (down).
- Bounce heights: $5 \times 0.8 = 4$, then each bounce is 80% of previous.
- Total distance = first drop + $2 \times$ sum of first 4 bounce heights (up and down).

$$5 + 2(4 + 3.2 + 2.56 + 2.048) = 5 + 2(11.808) = 28.616 \text{ m.}$$

5. Trigonometry

Key Concepts:

- Model periodic motion using sinusoidal functions.
- Know amplitude, period, phase shift, vertical shift.

Walkthrough Question 5:

A Ferris wheel of radius 15 m takes 40 seconds to complete a revolution. Bottom is 2 m above ground.

- (a) Write a sinusoidal function for rider height $h(t)$ at time t .
- (b) Find height after 25 seconds.

Solution:

- Center height = $15 + 2 = 17$ m.
- Angular velocity $\omega = \frac{2\pi}{40} = \frac{\pi}{20}$.
- Rider starts at bottom, so:

$$h(t) = 17 - 15 \cos\left(\frac{\pi}{20}t\right).$$

- Calculate height at $t = 25$:

$$h(25) = 17 - 15 \cos\left(\frac{25\pi}{20}\right) = 17 - 15 \cos\left(\frac{5\pi}{4}\right).$$

- $\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, so

$$h(25) = 17 + 15 \times \frac{\sqrt{2}}{2} \approx 27.61 \text{ m.}$$

6. Transformations

Key Concepts:

- Identify amplitude, period, phase shift, and vertical shift in trig functions.
- Period formula: Period = $\frac{2\pi}{|B|}$ for $y = A \sin(B(x - C)) + D$.

Walkthrough Question 6:

Given:

$$y = 2 \sin \left(3 \left(x - \frac{\pi}{4} \right) \right) + 1,$$

identify the transformations and sketch one period.

Solution:

- Amplitude = 2.
 - Period = $\frac{2\pi}{3}$.
 - Phase shift = $\frac{\pi}{4}$ right.
 - Vertical shift = +1.
 - Wave oscillates between -1 and 3, from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{4} + \frac{2\pi}{3}$.
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7. Conics

Key Concepts:

- Convert general form to standard form by completing the square.
- Identify ellipse, hyperbola, or parabola by equation form.
- Extract center, vertices, axes, and orientation.

Walkthrough Question 7:

Convert and identify the conic:

$$9x^2 + 16y^2 - 54x + 64y - 71 = 0.$$

Solution:

- Group terms:

$$9x^2 - 54x + 16y^2 + 64y = 71.$$

- Complete the square:

$$9(x^2 - 6x + 9) - 81 + 16(y^2 + 4y + 4) - 64 = 71,$$

$$9(x - 3)^2 + 16(y + 2)^2 = 216.$$

- Divide both sides:

$$\frac{(x - 3)^2}{24} + \frac{(y + 2)^2}{13.5} = 1.$$

- **Conclusion:** This is an ellipse centered at $(3, -2)$, with $a^2 = 24$, $b^2 = 13.5$.