Pre-Calculus 12 – Comprehensive Review Worksheet

Conics, Logs, Polynomials, Trig, Sequences, Transformations

Worksheet

Part A – Conics

1. Identify the type of conic represented by the equation:

$$9x^2 + 16y^2 - 144 = 0.$$

Then find the center, vertices, foci, and sketch the graph.

2. Write the equation of a parabola that has vertex at $(2, -3)$ and focus at $(2, 0)$.

3. Find the equation of the circle with center $(1, -2)$ and radius 5.

Part B – Logarithmic & Exponential Functions

4. Solve for x:

$$3^{2x+1} = 81.$$

5. Simplify the expression below as a single logarithm and write it in simplest form:

$$\log_5(x^2 - 1) - 2\log_5(x - 1).$$

6. Solve for x given:

$$\log_2(x + 4) + \log_2(x - 2) = 3.$$

State the domain restrictions.

Part C – Polynomial & Rational Functions

7. Use synthetic division to divide:

$$P(x) = 2x^5 - 3x^4 + 5x^3 - x + 7$$

by $x - 2$. Write the quotient and remainder.

8. Factor completely over the real numbers:

$$Q(x) = x^4 - 5x^3 + 6x^2 + 4x - 8.$$

9. Identify all vertical and horizontal asymptotes of the rational function:

$$R(x) = \frac{x^3 - 4x}{x^2 - x - 6}.$$

Part D – Trigonometry

10. Solve for all θ in degrees such that:

$$2\cos^2\theta - 3\cos\theta + 1 = 0, \quad 0^\circ \le \theta < 360^\circ.$$

11. Prove the trigonometric identity:

$$\frac{\sin x}{1 + \cos x} = \tan\frac{x}{2}.$$

Part E – Sequences & Series

12. The first term of an arithmetic sequence is 7 and the 10th term is 25. Find the common difference d and the sum of the first 15 terms.

13. A geometric sequence has first term $a_1 = 5$ and common ratio $r = \frac{1}{3}$. Find the 6th term and the sum of the first 8 terms.

Part F – Transformations of Functions

14. Describe fully the sequence of transformations applied to the parent function $f(x) = \sqrt{x}$ to obtain

$$g(x) = -2\sqrt{x - 3} + 5.$$

15. Sketch the graph of

$$h(x) = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1,$$

and state its amplitude, period, phase shift, and vertical shift.

Detailed Answer Key (Step-by-step)

Part A – Conics

1. Equation:

$$9x^2 + 16y^2 - 144 = 0 \implies 9x^2 + 16y^2 = 144.$$

Divide both sides by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

- This is an **ellipse** centered at the origin $(0, 0)$. - Semi-major axis: $a = \max(4, 3) = 4$ along x-axis. - Semi-minor axis: $b = 3$ along y-axis. - Vertices: $(\pm 4, 0)$. - Foci: distance c where $c^2 = a^2 - b^2 = 16 - 9 = 7$, so

$$c = \sqrt{7} \approx 2.645.$$

Foci are at $(\pm\sqrt{7}, 0)$.

Graph: ellipse centered at origin, wider along x-axis, touching $(\pm 4, 0)$ and $(0, \pm 3)$.

2. Parabola with vertex $(2, -3)$ and focus $(2, 0)$.

- Since vertex and focus share the same x-coordinate, parabola opens vertically. - Distance from vertex to focus: $p = 0 - (-3) = 3$ (positive means opens upward). - Equation form for vertical parabola:

$$(y - k) = \frac{1}{4p}(x - h)^2.$$

Substitute $h = 2$, $k = -3$, and $p = 3$:

$$y + 3 = \frac{1}{12}(x - 2)^2.$$

$$\boxed{y = -3 + \frac{1}{12}(x - 2)^2}.$$

3. Circle with center $(1, -2)$ and radius 5.

Standard form:

$$(x - h)^2 + (y - k)^2 = r^2,$$

so

$$\boxed{(x - 1)^2 + (y + 2)^2 = 25}.$$

Part B – Logarithmic & Exponential Functions

4. Solve:

$$3^{2x+1} = 81.$$

Note: $81 = 3^4$.

Set exponents equal:

$$2x + 1 = 4 \implies 2x = 3 \implies x = \frac{3}{2} = 1.5.$$

$$\boxed{x = 1.5}.$$

5. Simplify:

$$\log_5(x^2 - 1) - 2\log_5(x - 1).$$

Use log properties:
- Factor numerator: $x^2 - 1 = (x-1)(x+1)$, - $2\log_5(x-1) = \log_5(x-1)^2$.
Expression becomes:

$$\log_5[(x-1)(x+1)] - \log_5(x-1)^2 = \log_5 \frac{(x-1)(x+1)}{(x-1)^2}.$$

Simplify fraction:

$$= \log_5 \frac{x+1}{x-1}.$$

$$\boxed{\log_5\left(\frac{x+1}{x-1}\right)}.$$

Domain restrictions: $x > 1$ (to keep arguments positive).

6. Solve:

$$\log_2(x+4) + \log_2(x-2) = 3.$$

Combine logs:

$$\log_2[(x+4)(x-2)] = 3.$$

Rewrite exponential:

$$(x+4)(x-2) = 2^3 = 8.$$

Multiply:

$$x^2 + 4x - 2x - 8 = 8 \implies x^2 + 2x - 8 = 8.$$

Simplify:

$$x^2 + 2x - 16 = 0.$$

Use quadratic formula:

$$x = \frac{-2 \pm \sqrt{4+64}}{2} = \frac{-2 \pm \sqrt{68}}{2} = \frac{-2 \pm 2\sqrt{17}}{2} = -1 \pm \sqrt{17}.$$

Calculate approximate roots:

$$-1 + \sqrt{17} \approx -1 + 4.123 = 3.123,$$

$$-1 - \sqrt{17} \approx -1 - 4.123 = -5.123.$$

Check domain restrictions from logs:

$$x + 4 > 0 \implies x > -4, \quad x - 2 > 0 \implies x > 2.$$

Only $x \approx 3.123$ is valid.

$$\boxed{x = -1 + \sqrt{17} \approx 3.123}.$$

Part C – Polynomial & Rational Functions

7. Synthetic division of
$$2x^5 - 3x^4 + 5x^3 - x + 7$$
by $x - 2$.

Note: Write coefficients in order, including zeros for missing terms:
Coefficients: $2, -3, 5, 0, -1, 7$ (note zero for x^2 term).
Set up synthetic division with root 2:

$$\begin{array}{c|cccccc} 2 & 2 & -3 & 5 & 0 & -1 & 7 \\ & & 4 & 2 & 14 & 28 & 54 \\ \hline & 2 & 1 & 7 & 14 & 27 & 61 \end{array}$$

- Quotient polynomial:
$$2x^4 + x^3 + 7x^2 + 14x + 27,$$

- Remainder: 61.

$$\boxed{\begin{cases} \text{Quotient} = 2x^4 + x^3 + 7x^2 + 14x + 27, \\ \text{Remainder} = 61. \end{cases}}$$

8. Factor completely:
$$x^4 - 5x^3 + 6x^2 + 4x - 8.$$

Try rational roots using factors of 8: $\pm 1, \pm 2, \pm 4, \pm 8$.
Test $x = 1$:
$$1 - 5 + 6 + 4 - 8 = -2 \neq 0.$$

Test $x = 2$:
$$16 - 40 + 24 + 8 - 8 = 0.$$

So $(x - 2)$ is a factor.
Divide polynomial by $(x - 2)$:
Use synthetic division with 2:
Coefficients: $1, -5, 6, 4, -8$

$$\begin{array}{c|ccccc} 2 & 1 & -5 & 6 & 4 & -8 \\ & & 2 & -6 & 0 & 8 \\ \hline & 1 & -3 & 0 & 4 & 0 \end{array}$$

Quotient:
$$x^3 - 3x^2 + 0x + 4 = x^3 - 3x^2 + 4.$$

Now factor $x^3 - 3x^2 + 4$.
Try $x = 1$:
$$1 - 3 + 4 = 2 \neq 0,$$

Try $x = 2$:
$$8 - 12 + 4 = 0.$$

So $(x-2)$ is again a factor.
Divide x^3-3x^2+4 by $(x-2)$:
Coefficients: $1, -3, 0, 4$

$$\begin{array}{c|cccc} 2 & 1 & -3 & 0 & 4 \\ & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

Quotient:

$$x^2-x-2,$$

which factors as

$$(x-2)(x+1).$$

So complete factorization:

$$Q(x) = (x-2)^2(x-2)(x+1) = (x-2)^3(x+1).$$

$$\boxed{Q(x) = (x-2)^3(x+1).}$$

9. Rational function:

$$R(x) = \frac{x^3-4x}{x^2-x-6}.$$

Factor numerator and denominator:
Numerator:

$$x^3-4x = x(x^2-4) = x(x-2)(x+2).$$

Denominator:

$$x^2-x-6 = (x-3)(x+2).$$

Simplify:

$$R(x) = \frac{x(x-2)(x+2)}{(x-3)(x+2)} = \frac{x(x-2)(x+2)}{(x-3)(x+2)}.$$

Domain excludes $x=-2$ and $x=3$ (vertical asymptotes):
- $x=-2$ is a removable discontinuity (hole). - $x=3$ is a vertical asymptote.

Horizontal asymptote: degrees numerator $=3$, denominator $=2$, numerator degree higher by 1, so no horizontal asymptote but a slant (oblique) asymptote.

Find slant asymptote by polynomial division of numerator by denominator:
Divide numerator x^3-4x by denominator x^2-x-6:

$$\frac{x^3-4x}{x^2-x-6} = x+1+\frac{2x+6}{x^2-x-6}.$$

Slant asymptote is

$$y = x+1.$$

$$\boxed{\begin{cases} \text{Vertical asymptote at } x=3, \\ \text{Hole at } x=-2, \\ \text{Slant asymptote } y=x+1. \end{cases}}$$

Part D – Trigonometry

10. Solve:

$$2\cos^2\theta - 3\cos\theta + 1 = 0.$$

Let $u = \cos\theta$. Then

$$2u^2 - 3u + 1 = 0.$$

Use quadratic formula:

$$u = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}.$$

Solutions:

$$u_1 = \frac{3+1}{4} = 1, \quad u_2 = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}.$$

- $\cos\theta = 1$ gives $\theta = 0°$. - $\cos\theta = \frac{1}{2}$ gives $\theta = 60°$ and $300°$.

$$\boxed{\theta = 0°, 60°, 300°}.$$

11. Prove:

$$\frac{\sin x}{1+\cos x} = \tan\frac{x}{2}.$$

Use half-angle formulas:
Recall,

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}.$$

This is a known identity, but to verify:
Multiply numerator and denominator by $1 - \cos x$:

$$\frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} = \frac{\sin x(1-\cos x)}{1-\cos^2 x} = \frac{\sin x(1-\cos x)}{\sin^2 x}.$$

Simplify denominator:

$$= \frac{1-\cos x}{\sin x} = \frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}.$$

Thus,

$$\boxed{\frac{\sin x}{1+\cos x} = \tan\frac{x}{2}.}$$

Part E – Sequences & Series

12. Arithmetic sequence:
Given:

$$a_1 = 7, \quad a_{10} = 25.$$

Use formula:

$$a_n = a_1 + (n-1)d.$$

Find d:

$$25 = 7 + (10 - 1)d \implies 25 = 7 + 9d \implies 9d = 18 \implies d = 2.$$

Sum of first 15 terms:

$$S_{15} = \frac{15}{2}(a_1 + a_{15}),$$

Find a_{15}:

$$a_{15} = 7 + (15 - 1) \cdot 2 = 7 + 28 = 35.$$

Sum:

$$S_{15} = \frac{15}{2}(7 + 35) = \frac{15}{2}(42) = 15 \times 21 = 315.$$

$$\boxed{d = 2, \quad S_{15} = 315.}$$

13. Geometric sequence:
Given:

$$a_1 = 5, \quad r = \frac{1}{3}.$$

Find 6th term:

$$a_6 = a_1 r^5 = 5 \left(\frac{1}{3}\right)^5 = 5 \times \frac{1}{243} = \frac{5}{243}.$$

Sum of first 8 terms:

$$S_8 = a_1 \frac{1 - r^8}{1 - r} = 5 \frac{1 - \left(\frac{1}{3}\right)^8}{1 - \frac{1}{3}} = 5 \frac{1 - \frac{1}{6561}}{\frac{2}{3}} = 5 \times \frac{\frac{6560}{6561}}{\frac{2}{3}} = 5 \times \frac{6560}{6561} \times \frac{3}{2} = \frac{5 \times 6560 \times 3}{6561 \times 2}.$$

Simplify numerator:

$$5 \times 6560 \times 3 = 5 \times 19680 = 98400,$$

Denominator:

$$6561 \times 2 = 13122.$$

So,

$$S_8 = \frac{98400}{13122} \approx 7.5.$$

$$\boxed{a_6 = \frac{5}{243}, \quad S_8 \approx 7.5.}$$

Part F – Transformations of Functions

14. From

$$f(x) = \sqrt{x} \quad \to \quad g(x) = -2\sqrt{x-3} + 5.$$

Describe each transformation in order:

- Horizontal shift right by 3 (inside function $x - 3$), - Vertical stretch by factor of 2 (coefficient 2), - Reflection over x-axis (negative sign), - Vertical shift up by 5.

$$\boxed{\text{Shift right 3, stretch by 2, reflect over } x\text{-axis, shift up 5.}}$$

15. Graph of

$$h(x) = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1.$$

- Amplitude = coefficient before sine = 3. - Period = $\frac{2\pi}{|2|} = \pi$. - Phase shift = shift left by $\frac{\pi}{4}$ (inside the argument). - Vertical shift = down by 1.

$$\boxed{\begin{cases} \text{Amplitude} = 3, \\ \text{Period} = \pi, \\ \text{Phase shift} = -\frac{\pi}{4} \quad \text{(left)}, \\ \text{Vertical shift} = -1. \end{cases}}$$

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